
Contents

Part I Classical Geometry

1	Thales and Pythagoras	3
1.1	Thales' Theorem	3
1.2	Similar Figures	5
1.3	Properties of Angles	7
1.4	The Regular Pentagon	9
1.5	The Computation of Areas	11
1.6	A Remarkable Babylonian Document	13
1.7	The Pythagorean Theorem	14
1.8	Three Famous Problems of Greek Geometry	19
1.9	Exercises	21
2	The Elements of Euclid	27
2.1	Book I	29
2.2	Book III. Properties of Circles and Angles	39
2.3	Books V and VI. Real Numbers and Thales' Theorem	42
2.4	Books VII and IX. Number Theory	43
2.5	Book XI. Spatial Geometry and Solids	45
2.6	Book XII. Areas and Volumes of Circles, Pyramids, Cones and Spheres	48
2.7	Epilogue	52
2.8	Exercises	54
3	Conic Sections	61
3.1	The Parabola	62
3.2	The Ellipse	64
3.3	The Hyperbola	70
3.4	The Area of a Parabola	72
3.5	Exercises	73

4	Further Results in Euclidean Geometry	79
4.1	The Conchoid of Nicomedes, the Trisection of an Angle	79
4.2	The Archimedean Spiral	81
4.3	The Four Classical Centres of the Triangle	82
4.4	The Theorems of Menelaus and Ceva	87
4.5	The Theorems of Apollonius–Pappus–Stewart	89
4.6	The Euler Line and the Nine-Point Circle	91
4.7	Excircles and the Nagel Point	93
4.8	Miquel’s Theorems	94
4.9	Steiner’s Circle Theorems	98
4.10	Morley’s Theorem	104
4.11	Exercises	106
5	Trigonometry	113
5.1	Ptolemy and the Chord Function	113
5.2	Regiomontanus and Euler’s Trigonometric Functions	116
5.3	Arbitrary Triangles	119
5.4	Trigonometric Solution of Malfatti’s Problem	121
5.5	The Stereographic Projection	124
5.6	The Spherical Trigonometry of Right-Angled Triangles	127
5.7	The Spherical Trigonometry of General Triangles	133
5.8	The Area of a Spherical Triangle	139
5.9	Trigonometric Formulas for the Conics	140
5.10	The Great Discoveries of Kepler and Newton	141
5.11	Exercises	150

Part II Analytic Geometry

6	Descartes’ Geometry	159
6.1	The Principles of Descartes’ Geometry	159
6.2	The Regular Heptagon and Enneagon	162
6.3	The Trisection of an Angle and Cubic Equations	164
6.4	Regular Polygons in the Unit Circle	166
6.5	Van Roomen’s Famous Challenge	168
6.6	A Geometric Theorem of Fermat	170
6.7	Heron’s Formula for the Area of a Triangle	171
6.8	The Euler–Brahmagupta Formula for a Cyclic Quadrilateral	174
6.9	The Cramer–Castillon Problem	175
6.10	Exercises	178
7	Cartesian Coordinates	185
7.1	Equations of Lines and Circles	185
7.2	The Problem of Pappus	188
7.3	Conic Sections: Poles, Polars and Tangents	189

7.4	Problems of Minimum and Maximum	193
7.5	Some Famous Curves and Their Tangents	199
7.6	Curvature	210
7.7	The Euler Line by Euler	214
7.8	The Simson Line and Sturm's Circles	217
7.9	The Erdős–Mordell Inequality and the Steiner–Lehmus Theorem	222
7.10	The Butterfly	225
7.11	Thébault's Theorem	227
7.12	Billiards in an Ellipse	230
7.13	Urquhart's 'Most Elementary Theorem of Euclidean Geometry'	231
7.14	Exercises	232
8	To be Constructible, or not to be	241
8.1	The Complex Plane and the Logarithmic Spiral	242
8.2	Constructions with Ruler and Compass	246
8.3	The Method of Gauss and Vandermonde	247
8.4	The Regular 17-Sided Polygon	249
8.5	Constructions Impossible with Ruler and Compass	251
8.6	Exercises	254
9	Spatial Geometry and Vector Algebra	259
9.1	First Applications of Vectors	264
9.2	Gaussian Elimination, Volume and Determinant	265
9.3	Norm and Scalar Product	268
9.4	The Outer Product	270
9.5	Spherical Trigonometry Revisited	274
9.6	Pick's Theorem	277
9.7	A Theorem on Pentagons in Space	280
9.8	Archimedean Solids	282
9.9	Exercises	287
10	Matrices and Linear Mappings	291
10.1	Changes of Coordinates	291
10.2	Linear Mappings	292
10.3	Gram's Determinant	296
10.4	Orthogonal Mappings and Isometries	300
10.5	Skew-Symmetric Matrices, the Cayley Transform	304
10.6	Eigenvalues and Eigenvectors	307
10.7	Quadratic Forms	308
10.8	Exercises	313

11 Projective Geometry	319
11.1 Perspective and Central Projection	320
11.2 Poncelet's Principle of Central Projection.....	323
11.3 The Projective Line.....	330
11.4 The Projective Plane	335
11.5 The Principle of Duality.....	338
11.6 The Projective Theory of Conics	340
11.7 Exercises	342
12 Solutions to the Exercises	345
References	403
Figure Sources and Copyright	419