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Introduction

A few decades ago I learned about degree 3 spline functions and wrote programs that use them to pass a curve through given points in a plane. I've been waiting for such curves to be implemented in software like Inkscape and FreeCAD for a long time.
Freya Holmér
's excellent video on Youtube called: “The continuity of Splines. »
See: https://www.youtube.com/watch?v=jvPPXbo87ds
by Freya Holmér see: https://www.youtube.com/@Acegikmo

Unfortunately, the curve I would name "Math-spline", based on the spline functions that I describe later and that I implemented in the following web page:
https://www.juggling.ch/gisin/bgweb/aprod2000_perso/spline_curve_math.html
is not described by Freya Homér nor used in the software I know.

Here are some characteristics of this “Math-spline” curve:
° continuous
° of continuously varying tangents along the curve
° of radii of curvature varying continuously along the curve, it is G^2
° the influence of the points is practically on the 8 neighboring segments of the point
° is easy to calculate, quickly
° can easily be closed
° can have breakpoints, so where the tangent varies discontinuously.
° is invariant under rotation, symmetry and dilation

We will suppose to have a list of n = nbPts points $\vec{v}_i = (x_i; y_i)$ in the plane, for i = 1.. nbPts
We want to pass a “natural” curve through these points. To do this, we will define the intervals h_i = distance between $\vec{v}_i$ and $\vec{v}_{i+1} = \|\vec{v}_{i+1} - \vec{v}_i\|$ and the times $t_i = t_{i-1} + h_i$, with $t_1 = 0$ and we will determine two "spline functions" determined by the data: (t_i; x_i) and (t_i; y_i), for i = 1. .nbPts.
We will see two ways of calculating these “spline functions”, how to make them periodic, how to introduce breaks in the curve and that they are invariant under rotation, symmetry and dilation.

Mathematics of "spline function" calculation

Reference : http://www.unige.ch/~hairer/polycop.html

" Handouts of the course "Numerical Analysis" (June 2005), Chapter II. Interpolation and Approximation » pages 46 to 52. (http://www.unige.ch/~hairer/poly/chap2.pdf)

This is the first way I learned, but there is another one that I prefer. It is described further on page 11 and is the one I use in the program.

Data :

(t_i; y_i), for i = 1.. nbPts

Spline passing through the points, cubic by piece, twice continuously differentiable:

$s(t_i)=y_i$, for i = 1.. nbPts

$h_i=t_{i+1}-t_i \quad \delta_i=\frac{y_{i+1}-y_i}{t_{i+1}-t_i}=\frac{y_{i+1}-y_i}{h_i}$, for i = 1.. nbPts - 1

For i = 1.. nbPts - 1, $t \in [t_i .. t_{i+1}]$

$$s_i(t)=y_i+(t-t_i)\cdot\delta_i+\frac{(t-t_i)\cdot(t-t_{i+1})}{h_i^2}\cdot[(p_i-\delta_i)\cdot(t-t_{i+1})+(p_{i+1}-\delta_i)\cdot(t-t_i)]$$

Where p i is the slope of the spline at $t = t_i$.

So $s'(t_i)=p_i$.

So $s'_i(t_i)=p_i$ and $s'_i(t_{i+1})=p_{i+1}=s'_{i+1}(t_{i+1})$.

Verifications:

$s_i(t_i)=y_i; s_i(t_{i+1})=y_{i+1}$

$$s_i(t_i+\epsilon)=y_i+\epsilon\cdot\delta_i+\frac{\epsilon\cdot(-h_i)}{h_i^2}\cdot[p_i\cdot(-h_i)-\delta_i\cdot(-h_i)+(..)\cdot\epsilon]$$

$s_i(t_i+\epsilon)=y_i+\epsilon\cdot p_i$, with ϵ^2 negligible, so p_i is indeed the slope at $t = t_i$.

$$s_i(t_{i+1}-\epsilon)=y_i+(h_i-\epsilon)\cdot\delta_i+\frac{h_i\cdot(-\epsilon)}{h_i^2}\cdot[(..)\cdot\epsilon+p_{i+1}\cdot(h_i-\epsilon)-\delta_i\cdot(h_i-\epsilon)]$$

$s_i(t_{i+1}-\epsilon)=y_{i+1}-\epsilon\cdot p_{i+1}$, with ϵ^2 negligible, so p_{i+1} is indeed the slope at $t = t_{i+1}$.

Condition to be met for the second derivative to be continuous:

$s''_{i-1}(t_i)=\frac{2}{h_{i-1}}\cdot[2p_i+p_{i-1}-3\cdot\delta_{i-1}]$ and $s''_i(t_i)=\frac{2}{h_i}\cdot[3\cdot\delta_i-(p_{i+1}+2p_i)]$

It is necessary that : $s''_{i-1}(t_i)=s''_i(t_i)$

So : $\frac{2}{h_{i-1}}\cdot[2p_i+p_{i-1}-3\cdot\delta_{i-1}]=\frac{2}{h_i}\cdot[3\cdot\delta_i-(p_{i+1}+2p_i)]$

Multiply by $\frac{h_{i-1}\cdot h_i}{2}$ and rearrange the terms:

$h_i\cdot p_{i-1}+2\cdot(h_{i-1}+h_i)\cdot p_i+h_{i-1}\cdot p_{i+1}=3\cdot(h_{i-1}\cdot\delta_i+h_i\cdot\delta_{i-1})$

Condition to be fulfilled for i = 2.. nbPts - 1.

For “Natural Splines”, we have: $s''_1(t_1)=s''_{n-1}(t_n)=0$, so

$2p_1+p_2=3\cdot\delta_1$, or also : $2h_0\cdot p_1+h_0\cdot p_2=3\cdot h_0\cdot\delta_1$ and

$p_{n-1}+2p_n=3\cdot\delta_{n-1}$, or also : $2h_n\cdot p_{n-1}+h_n\cdot p_n=3\cdot h_n\cdot\delta_{n-1}$ h_0 and h_n are arbitrary nonzero.

System of equations to solve to obtain the value of p_i $i = 1..n$
n = number of points = nbPts

$$\begin{pmatrix} 2\cdot h_0 & h_0 & 0 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot h_2+2\cdot h_1 & h_1 & 0 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot h_3+2\cdot h_2 & h_2 & 0 & 0 & 0 \\ 0 & 0 & h_4 & 2\cdot h_4+2\cdot h_3 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_5 & 2\cdot h_5+2\cdot h_4 & h_4 & 0 \\ 0 & 0 & 0 & 0 & h_{n-1} & 2\cdot h_{n-1}+2\cdot h_{n-2} & h_{n-2} \\ 0 & 0 & 0 & 0 & 0 & h_n & 2\cdot h_n \end{pmatrix} \circ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_{n-1} \\ p_n \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_{n-1} \\ q_n \end{pmatrix}$$

n = 7 in the matrix example above which is an n x n *matrix*

$h_0 = h_n = 1; h_i = t_{i+1} - t_i \quad \delta_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = \frac{y_{i+1} - y_i}{h_i}$, for i = 1.. nbPts - 1

$q_i = 3\cdot(h_i\cdot\delta_{i-1} + h_{i-1}\cdot\delta_i)$ i =2..n - 1

$q_1 = 3\cdot h_0\cdot\delta_1$ and $q_n = 3\cdot h_n\cdot\delta_{n-1}$.

We have used the case of the "natural" spline, which is defined by the characteristic that its second derivative at the edges is zero.

<u>Resolution :</u>

$h_0 = 1$; $h_n = 1$($h_0 = \infty$ and $h_n = \infty$ would be natural)

$diag_1 = 2\cdot h_0$; $diag_n = 2\cdot h_n$;

$diag_i = 2\cdot h_i + 2\cdot h_{i-1}$ If $diag_i = 0$, then $diag_i = 1$ i =2..n - 1

$lft_i = h_i$ i = 0..n

for i=2 to n do $diag_i = diag_i - \frac{lft_i}{diag_{i-1}}\cdot lft_{i-2}$ and $q_i = q_i - \frac{lft_i}{diag_{i-1}}\cdot q_{i-1}$ we have: $lft_{i-2} = \text{right}_{i-1}$

$$p_n = \frac{q_n}{diag_n}$$

for i = n -1 downto 1 do $p_i = \frac{q_i - lft_{i-1}\cdot p_{i+1}}{diag_i}$ we have: $lft_{i-1} = \text{right}_i$

<u>Change of parameterization of the spline function.</u>

For i = 1.. nbPts - 1, $t \in [t_i..t_{i+1}]$ $t = t_i + \tau\cdot h_i$ $\tau \in [0..1]$

$s_i(t_i + \tau\cdot h_i) = y_i + \tau\cdot(y_{i+1} - y_i) +$ ¿

$\tau\cdot(\tau - 1)\cdot[(p_i\cdot h_i - y_{i+1} + y_i)\cdot(\tau - 1) + (p_{i+1}\cdot h_i - y_{i+1} + y_i)\cdot\tau]$

$s_i(t_i + \tau\cdot h_i) = y_i + b_i\cdot\tau + c_i\cdot\tau^2 + d_i\cdot\tau^3$

$b_i = p_i\cdot h_i$

$c_i = 3\cdot(y_{i+1} - y_i) - (p_{i+1} + 2\cdot p_i)\cdot h_i$

$d_i = (p_{i+1} + p_i)\cdot h_i - 2\cdot(y_{i+1} - y_i)$

Case of the "spline function" having the slopes at the edges defined

We assume given the slopes at the edges: p_1= given; p_n= given,

$$\begin{pmatrix} 2\cdot h_2+2\cdot h_1 & h_1 & 0 & 0 & 0 & 0 \\ h_3 & 2\cdot h_3+2\cdot h_2 & h_2 & 0 & 0 & 0 \\ 0 & h_4 & 2\cdot h_4+2\cdot h_3 & h_3 & 0 & 0 \\ 0 & 0 & h_5 & 2\cdot h_5+2\cdot h_4 & h_4 & 0 \\ 0 & 0 & 0 & h_{n-2} & 2\cdot h_{n-2}+2\cdot h_{n-3} & h_{n-3} \\ 0 & 0 & 0 & 0 & h_{n-1} & 2\cdot h_{n-1} \end{pmatrix} \circ \begin{pmatrix} p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_{n-2} \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_{n-2} \\ q_{n-1} \end{pmatrix}$$

n = 8 in the matrix example above

$q_i=3\cdot(h_i\cdot\delta_{i-1}+h_{i-1}\cdot\delta_i)$ i =3..n - 2

$q_2=3\cdot(h_2\cdot\delta_1+h_1\cdot\delta_2)-h_2\cdot p_1$ and $q_{n-1}=3\cdot(h_{n-1}\cdot\delta_{n-2}+h_{n-2}\cdot\delta_{n-1})-h_{n-2}\cdot p_n$

For a **periodic spline** , the easiest way is to increase the matrix by 10 points, which merge with the points to give a closed curve which intersects, then calculate the *pi* , *then* eliminate the first 5 and the last 5.

The case of the periodic spline is taken up later, after having seen another approach to calculating the spline, which will make the task easier.

Study of the case where an interval is of zero length, first approach.

Further after determining another way to calculate the "spline function", the case where the interval is of zero length is restated and is simpler. So the following can be skipped.

Take the case where $h_4=0$, in this case the matrix becomes:

$$\begin{pmatrix} 2\cdot h_2 & h_2 & 0 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot h_2+2\cdot h_1 & h_1 & 0 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot h_3+2\cdot h_2 & h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot h_3 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_5 & 2\cdot h_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{n-1} & 2\cdot h_{n-1}+2\cdot h_{n-2} & h_{n-2} \\ 0 & 0 & 0 & 0 & 0 & h_{n-2} & 2\cdot h_{n-2} \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_n \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_n \end{pmatrix}$$

$q_4=3\cdot h_3\cdot \delta_4andq_5=3\cdot h_5\cdot \delta_4$

In this case, lines 4 and 5 come down to a system of two equations with two unknowns:

$2\cdot p_4+p_5=3\cdot \delta_4$and
$p_4+2\cdot p_5=3\cdot \delta_4$

Workaround: $p_4=p_5=\delta_4=\dfrac{y_5-y_4}{0}$.

If $y_4=y_5$, the value of $\delta_4$is undetermined.

By setting it equal to 0 ($\delta_4=0$), we have the advantage that the segment going from t_4 to t_5 remains constant and that a break can occur in this segment of zero and constant length, merged with the points $y_4=y_5$.

On the other hand, since the values of $p_4$and $p_5$are known, the matrix breaks down into two independent matrices. It is practically useless, but it had to be noticed.

Let's remember that :

$s_i(t_i+\tau\cdot h_i)=y_i+b_i\cdot\tau+c_i\cdot\tau^2+d_i\cdot\tau^3$

$b_i=p_i\cdot h_i$

$c_i=3\cdot(y_{i+1}-y_i)-(p_{i+1}+2\cdot p_i)\cdot h_i$

$d_i=(p_{i+1}+p_i)\cdot h_i-2\cdot(y_{i+1}-y_i)$

So $s_4(t_4+\tau\cdot 0)=y_4+0\cdot\tau+0\cdot\tau^2+0\cdot\tau^3=y_4$= constant

$s''_3(t_4)=\dfrac{2}{h_3}\cdot[2p_4+p_3-3\cdot\delta_3]=0 \Rightarrow p_3+2p_4=3\cdot\delta_3$

$s''_5(t_5)=\dfrac{2}{h_5}\cdot[3\cdot\delta_5-(p_6+2p_5)]=0 \Rightarrow 2p_5+p_6=3\cdot\delta_5$

Lines 4 and 5 can be replaced by the lines above, to modify the system and have natural boundary conditions. This will be easier with the other way of determining the “spline function”.

Question, is the spline dependent on the choice of reference frame?

In other words,
if we apply a linear transformation to the points defining the spline, then we determine the spline from these points or
if we determine the spline from the points, then we apply the same linear transformation to it, obtains- do we have the same curve?
The following shows that yes if the transformation is a rotation, a symmetry or a dilation.
This was also checked programmatically.

Note: $\begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}$ the 2x2 linear transformation matrix.

It will be necessary to imagine two cases, one where the transformation is orthogonal, that is to say that it preserves the distances and the general case.

Note: the $\left(\vec{V}\right)=\begin{pmatrix} x_1 & y_1 \\ x_i & y_i \\ .. & .. \\ x_n & y_n \end{pmatrix}$ *n x 2* matrix of points defining the spline.

Note: $\left(\vec{S}(t)\right)$= the ($\begin{pmatrix} Sx_1(t) & Sy_1(t) \\ Sx_i(t) & Sy_i(t) \\ .. & .. \\ Sx_{n-1}(t) & Sy_{n-1}(t) \end{pmatrix}$ *n -1) x 2* matrix defining the curve of the spline.

$\left(\vec{S}(t)\right)$ is a function *F* of *t* and of $\left(\vec{V}\right)$. It will be developed further.

Note: $\widetilde{\left(\vec{S}(t)\right)}=\left(\vec{S}(t)\right)\circ\begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}$= the ($\begin{pmatrix} \widetilde{Sx_1}(t) & \widetilde{Sy_1}(t) \\ \widetilde{Sx_i}(t) & \widetilde{Sy_i}(t) \\ .. & .. \\ \widetilde{Sx_{n-1}}(t) & \widetilde{Sy_{n-1}}(t) \end{pmatrix}$ *n -1) x 2* matrix defining the curve of the spline after its linear transformation.

Note: the $\widetilde{\left(\vec{V}\right)}=\left(\vec{V}\right)\circ\begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} \widetilde{x_1} & \widetilde{y_1} \\ \widetilde{x_i} & \widetilde{y_i} \\ .. & .. \\ \widetilde{x_n} & \widetilde{y_n} \end{pmatrix}$ *n x 2* matrix of points defining the spline, after linear transformation.

The question is to know if $\widetilde{\left(\vec{S}(t)\right)}$ is obtained by the same function *F* of *t* and of $\widetilde{\left(\vec{V}\right)}$?

Let us explain the function F.

$$(\vec{S}(t)) = (\vec{A}) + (\vec{B})\cdot t + (\vec{C})\cdot t^2 + (\vec{D})\cdot t^3$$

$(\vec{A}) = \begin{pmatrix} x_1 & y_1 \\ x_i & y_i \\ .. & .. \\ x_{n-1} & y_{n-1} \end{pmatrix}$ and $(\widetilde{\vec{A}}) \overset{déf.}{=} \begin{pmatrix} \widetilde{x_1} & \widetilde{y_1} \\ \widetilde{x_i} & \widetilde{y_i} \\ .. & .. \\ \widetilde{x_{n-1}} & \widetilde{y_{n-1}} \end{pmatrix}$. $(\widetilde{\vec{A}})$ is the vector obtained from the TL of the points.

We saw on the previous page that: $(\widetilde{\vec{A}}) = (\vec{A}) \circ \begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}$, because $(\widetilde{\vec{V}}) = (\vec{V}) \circ \begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}$

$$(\vec{B}) = \begin{pmatrix} px_1\cdot h_1 & py_1\cdot h_1 \\ px_i\cdot h_i & py_i\cdot h_i \\ .. & .. \\ px_{n-1}\cdot h_{n-1} & py_{n-1}\cdot h_{n-1} \end{pmatrix} == \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_i & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & h_{n-1} \end{pmatrix} \circ \begin{pmatrix} px_1 & py_1 \\ px_i & py_i \\ .. & .. \\ px_{n-1} & py_{n-1} \end{pmatrix} \_ (H\,diag) \circ \begin{pmatrix} px_1 & py_1 \\ px_i & py_i \\ .. & .. \\ px_{n-1} & py_{n-1} \end{pmatrix}$$

$\begin{pmatrix} px_1 & py_1 \\ px_i & py_i \\ .. & .. \\ px_n & py_n \end{pmatrix} = (H\,1)^{-1} \circ \begin{pmatrix} qx_1 & qy_1 \\ qx_i & qy_i \\ .. & .. \\ qx_n & qy_n \end{pmatrix}$ where is the large $(H\,1)$ n x n matrix from page 2.

$\begin{pmatrix} qx_1 & qy_1 \\ qx_i & qy_i \\ .. & .. \\ qx_n & qy_n \end{pmatrix} = (H\,2) \circ \begin{pmatrix} x_1 & y_1 \\ x_i & y_i \\ .. & .. \\ x_n & y_n \end{pmatrix}$ where

$(H\,2)$ n x n matrix defining the values of the q_i which are defined on page 2.
It only depends on the values of h_i, which are either independent of the points, or only dependent on the distances between the points.

Result :

$$(\vec{B}) == (H\,diag) \circ (H1)^{-1} \circ (H2\,diag) \circ \begin{pmatrix} x_1 & y_1 \\ x_i & y_i \\ .. & .. \\ x_n & y_n \end{pmatrix} \_ (H\,diag) \circ (H\,1)^{-1} \circ (H2\,diag) \circ (\vec{A})$$

and if $(\widetilde{\vec{B}})$ is the vector obtained from the TL of the points, then

$$(\widetilde{\vec{B}}) \overset{déf.}{=} (H\,diag) \circ (H\,1)^{-1} \circ (H\,2\,diag) \circ \begin{pmatrix} \widetilde{x_1} & \widetilde{y_1} \\ \widetilde{x_i} & \widetilde{y_i} \\ .. & .. \\ \widetilde{x_n} & \widetilde{y_n} \end{pmatrix} = (H\,diag) \circ (H1)^{-1} \circ (H2\,diag) \circ (\widetilde{\vec{A}})$$

$$= (H\,diag) \circ (H1)^{-1} \circ (H2\,diag) \circ (\vec{A}) \circ \begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix} = (\vec{B}) \circ \begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we have the relation : $(\widetilde{\vec{B}}) = (\vec{B}) \circ \begin{pmatrix} 2x2 & 0 \\ 0 & 0 \end{pmatrix}$!!!

We can do the same to show that:
if $\left(\widetilde{\vec{C}}\right)$ and $\left(\widetilde{\vec{D}}\right)$ are the vectors obtained from the TLs of the points, then
$\left(\widetilde{\vec{C}}\right)=\left(\vec{C}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}$ and $\left(\widetilde{\vec{D}}\right)=\left(\vec{D}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}$!!!

This allows us to obtain:
The linear transformation of the spline:
$$\left(\vec{S}(t)\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}=\left(\vec{A}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}+\left(\vec{B}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}\cdot t+\left(\vec{C}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}\cdot t^2+\left(\vec{D}\right)\circ\begin{pmatrix}2x2 & 0\\ 0 & 0\end{pmatrix}\cdot t^3$$
$$\left(\widetilde{\vec{A}}\right)+\left(\widetilde{\vec{B}}\right)\cdot t+\left(\widetilde{\vec{C}}\right)\cdot t^2+\left(\widetilde{\vec{D}}\right)\cdot t^3=\left(\widetilde{\vec{S}(t)}\right)$$
is equal to the spline obtained from the points which have undergone the linear transformation.

This shows that if the matrices $(H1)$; $(H2)$; $(H\,diag)$; ...
are independent of the points, then the curve of the spline is independent of the choice of reference frame, because to make the points undergo a linear transformation then calculate the spline or calculate the spline then make it undergo the linear transformation is the same thing.

In the case where we choose $h_i=1$ for all i, we do have independence at the points.

In the case where one chooses h_i=distance between the point p_i and the point p_i+1, the independence is true only if the linear transformation is orthogonal, for example a rotation or a symmetry.

A first summary:

The spline curve defined by the previous calculations is:
° continuous
° of continuously varying tangents along the curve
° of radius of curvature varying continuously along the curve
° the influence of the points is practically on the 8 neighboring segments of the point
° is easy to calculate, quickly
° can easily be closed
° can have breakpoints, so where the tangent varies discontinuously.

Some justifications for the above statements.
Between two points, the coordinates of the points of the spline are given by a cubic polynomial, so between two points the curve is perfectly smooth, even from the mathematical point of view.

It remains to see that the statements are correct at the points of definition of the spline.
The continuity is obvious, because the spline passes through the given points.

The way to construct the spline curve involves the construction of two functions that are twice continuously differentiable, sX(t) and sY(t), which define the parametric equation of the spline.
(X; Y) = (sX(t); sY(t)) for t varying from 0 to t_{nbPts} .

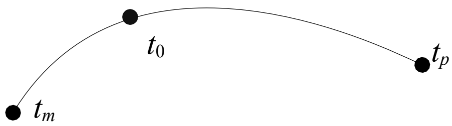
Slope of the tangent at $t = \frac{sY'(t)}{sX'(t)}$, which varies continuously.
When the denominator is zero, the slope is simply vertical.

The radius of curvature at $t = \frac{\left(sX'^2(t)+sY'^2(t)\right)^{\frac{3}{2}}}{\left|sX'(t)\cdot sY''(t)+sX''(t)\cdot sY'(t)\right|}$, which varies continuously.
When the denominator is zero, the radius of curvature is infinite.
It varies continuously, because the two coordinates are twice continuously differentiable.

Case G^1 where the existence of the second derivative is not desired.

To simplify, we may not want the curve to be continuously differentiable twice, but just continuously differentiable once. We just have to choose reasonable values of p_i.

We want p_i to approximate "at best" the derivative of the curve at t_i.



Let f *be* a $C^{2\text{ function}}$, hence twice continuously differentiable on the interval $[t_m; _p]$ _
$t_m < t_0 < t_p$ given.
With: $h_p = t_p - t_0$ and $h_m = t_0 - t_m$, both positive.

Knowing $f(t_m)$; $f(t_0)$ and $f(t_p)$, we would like to approximate " at best " $f'(t_0)$.

We have :

$$f(t_p) = f(t_0) + f'(t_0) \cdot h_p + f''(\tau_p) \cdot \frac{h_p}{2} \text{ where } \tau_p \in [t_0; t_p]$$

$$f(t_m) = f(t_0) - f'(t_0) \cdot h_m + f''(\tau_m) \cdot \frac{h_m}{2} \text{ where } \tau_m \in [t_m; t_0]$$

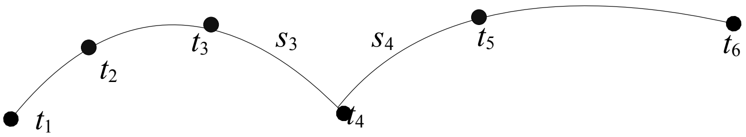
$$\frac{\alpha \cdot \frac{f(t_p) - f(t_0)}{h_p} + \beta \cdot \frac{f(t_0) - f(t_m)}{h_m}}{\alpha + \beta} = f'(t_0) + \frac{\alpha \cdot h_p \cdot f''(\tau_p) - \beta \cdot h_m \cdot f''(\tau_m)}{2 \cdot (\alpha + \beta)}$$

We can think that $f''(\tau_m)$ and $f''(\tau_p)$ are close, we don't have better, so a choice that seems reasonable to me is: $\alpha = h_m$ and $\beta = h_p$.

So we approximate $f'(t_0)$ by: $\dfrac{h_m \cdot \frac{f(t_p) - f(t_0)}{h_p} + h_p \cdot \frac{f(t_0) - f(t_m)}{h_m}}{h_m + h_p}$

A less good, but simpler choice is: $\alpha = h_p$ and $\beta = h_m$. which approximates $f'(t_0)$ by $\dfrac{f(t_p) - f(t_m)}{t_p - t_m}$.

Case G^1 and with discontinuity of a tangent.



If there is a break at t_4, s_3 only depends on the points at t_2, t_3, and t_4, but not at t_5.
If there is a break at t_4, s_4 only depends on the points at t_4, t_5, and t_6, but not at t_3.
(If there is a break at t_3, s_3 only depends on the points at t_3, t_4, and t_5, but not at t_2.)

In the case of a break in t_4.

We would like to have $s''_3(t_4)=0$
$\frac{1}{2}\cdot s''_3(t_4)=c_3+3\cdot d_3=h_3\cdot(p_3+2p_4-3\delta_3)$, So : $p_4=\frac{3\delta_3-p_3}{2}$
Here we refer to p_4 of s_3, which is different from p_4 of s_4.

We would like to have $s''_4(t_4)=0$
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Here we refer to p_4 of s_4, which is different from p_4 of s_3.

In the case of a break at t_3.

We would like to have $s''_3(t_3)=0$
$\frac{1}{2}\cdot s''_4(t_3)=c_3=h_3\cdot(3\delta_3-p_4-2p_3)$, So : $p_3=\frac{3\delta_3-p_4}{2}$
Here we refer to the p_3 of s_3, which is different from the p_3 of s_2.

Another way of approaching the calculation of the spline function, M_i = second derivative.
Reference: "Introduction to numerical analysis" by J. Stoer and R. Bulirsch, Springer-Verlag 1983.
ISBN: 0-387-90420-4 New York or 3-540-90420-4 Berlin Heidelberg.
This way has advantages over the one seen in the previous pages.
Data :
$(t_i ; y_i)$, for i = 1.. nbPts

Spline passing through the points, cubic by piece, twice continuously differentiable:
$s(t_i)=y_i$, for i = 1.. nbPts

$h_i=t_{i+1}-t_i \quad \delta_i=\frac{y_{i+1}-y_i}{t_{i+1}-t_i}=\frac{y_{i+1}-y_i}{h_i}$, for i = 1.. nbPts - 1

For i = 1.. nbPts - 1, $t \in [t_i .. t_{i+1}]$

$$s_i(t)=\frac{(t_{i+1}-t)^3}{6h_i}\cdot M_i+\frac{(t-t_i)^3}{6h_i}\cdot M_{i+1}+\frac{(t_{i+1}-t)}{h_i}\cdot\left(y_i-\frac{h_i^2}{6}\cdot M_i\right)+\frac{(t-t_i)}{h_i}\cdot\left(y_{i+1}-\frac{h_i^2}{6}\cdot M_{i+1}\right)$$

Where M i is the second derivative of the spline $_{at}$ $t = t_i$.
So $s''_i(t_i)=M_i$ and $s''_i(t_{i+1})=M_{i+1}=s''_{i+1}(t_{i+1})$.
So the second derivative of s(t) is continuous, provided that the derivative is continuous.

$$s'_i(t)=-\frac{(t_{i+1}-t)^2}{2h_i}\cdot M_i+\frac{(t-t_i)^2}{2h_i}\cdot M_{i+1}+\frac{-1}{h_i}\cdot\left(y_i-\frac{h_i^2}{6}\cdot M_i\right)+\frac{1}{h_i}\cdot\left(y_{i+1}-\frac{h_i^2}{6}\cdot M_{i+1}\right)$$

At the edges:

$$s'_{i-1}(t_i)=\frac{h_{i-1}}{2}\cdot M_i-\frac{1}{h_{i-1}}\cdot\left(y_{i-1}-\frac{h_{i-1}^2}{6}\cdot M_{i-1}\right)+\frac{1}{h_{i-1}}\cdot\left(y_i-\frac{h_{i-1}^2}{6}\cdot M_i\right)=\frac{h_{i-1}}{6}\cdot(2M_i+M_{i-1})+\frac{y_i-y_{i-1}}{h_{i-1}}$$

$$s'_i(t_i)=-\frac{h_i}{2}\cdot M_i-\frac{1}{h_i}\cdot\left(y_i-\frac{h_i^2}{6}\cdot M_i\right)+\frac{1}{h_i}\cdot\left(y_{i+1}-\frac{h_i^2}{6}\cdot M_{i+1}\right)=\frac{h_i}{6}\cdot(-2M_i-M_{i+1})+\frac{y_{i+1}-y_i}{h_i}$$

Condition for the derivative to be continuous: $s'_{i-i}(t_i)=s'_i(t_i)$, therefore:

$h_{i-1}\cdot(2M_i+M_{i-1})+6\cdot\delta_{i-1}=h_i\cdot(-2M_i-M_{i+1})+6\cdot\delta_i$

What becomes by rearranging the terms:

$h_{i-1}\cdot M_{i-1}+2(h_{i-1}+h_i)\cdot M_i+h_i\cdot M_{i+1}=6\cdot(\delta_i-\delta_{i-1})$ for i = 2.. nbPts - 1

Change of parameterization of the curve.
For i = 1.. nbPts - 1, $t \in [t_i .. t_{i+1}]$ $\quad t=t_i+\tau\cdot h_i \quad \tau\in[0..1]$

$$s_i(t_i+\tau\cdot h_i)=(1-\tau)^3\cdot\frac{h_i^2}{6}\cdot M_i+\tau^3\cdot\frac{h_i^2}{6}\cdot M_{i+1}+(1-\tau)\cdot\left(y_i-\frac{h_i^2}{6}\cdot M_i\right)+\tau\cdot\left(y_{i+1}-\frac{h_i^2}{6}\cdot M_{i+1}\right)$$

$$s_i(t_i+\tau\cdot h_i)=y_i+b_i\cdot\tau+c_i\cdot\tau^2+d_i\cdot\tau^3$$

$b_i=y_{i+1}-y_i-\frac{2M_i+M_{i+1}}{6}\cdot h_i^2$ before: $b_i=p_i\cdot h_i$, so: $p_i=\delta_i-\frac{2M_i+M_{i+1}}{6}\cdot h_i$

$c_i=\frac{h_i^2}{2}\cdot M_i$ before: $c_i=3\cdot(y_{i+1}-y_i)-(p_{i+1}+2\cdot p_i)\cdot h_i$, so: $M_i=6\cdot\frac{\delta_i}{h_i}-\frac{(2\cdot p_{i+1}+4\cdot p_i)}{h_i}$

$d_i=\frac{h_i^2}{6}\cdot(M_{i+1}-M_i)$ before : $d_i=(p_{i+1}+p_i)\cdot h_i-2\cdot(y_{i+1}-y_i)$

In matrix form, this becomes

System of equations to be solved to obtain the value of M_i i =2..n - 1

n = number of points = nbPts

$M1 = Mn = 0$

These two equalities correspond to the case of the "natural" spline, which is defined by the characteristic that its second derivative at the edges is zero.

$$\begin{pmatrix} 2\cdot(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot(h_2+h_3) & h_3 & 0 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot(h_3+h_4) & h_4 & 0 & 0 & 0 \\ 0 & 0 & h_4 & 2\cdot(h_4+h_5) & h_5 & 0 & 0 \\ 0 & 0 & 0 & h_5 & 2\cdot(h_5+h_6) & h_6 & 0 \\ 0 & 0 & 0 & 0 & h_{n-3} & 2\cdot(h_{n-3}+h_{n-2}) & h_{n-2} \\ 0 & 0 & 0 & 0 & 0 & h_{n-2} & 2\cdot(h_{n-2}+h_{n-1}) \end{pmatrix} \circ \begin{pmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_{n-1} \end{pmatrix} = \begin{pmatrix} r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_{n-1} \end{pmatrix}$$

n = 9 in the matrix example above

$h_i = t_{i+1} - t_i \quad \delta_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = \frac{y_{i+1} - y_i}{h_i}$, for i = 1.. nbPts - 1

$\delta_i = 0$ if the numerator is zero, even if the denominator is also zero. It's arbitrary, but practical.

$r_i = 6\cdot(\delta_i - \delta_{i-1}) \quad i$ =2..n - 1

Resolution :

$diag_i = 2\cdot(h_{i-1}+h_i)$; If $diag_i = 0$, then $diag_i = 1$ i = 2.. n -1 The first line is i =2

$lft_i = h_{i-1}$ i = 1.. n -1

for i =3 to n -1 do $diag_i = diag_i - \frac{lft_i}{diag_{i-1}} \cdot lft_i$ and $r_i = r_i - \frac{lft_i}{diag_{i-1}} \cdot r_{i-1}$ we have: $lft_i = \text{right}_{i-1}$

$$M_{n-1} = \frac{r_{n-1}}{diag_{n-1}}$$

for $i = n$ -2 downto 2 do $M_i = \frac{r_i - lft_{i+1} \cdot M_{i+1}}{diag_i}$ we have: $lft_{i+1} = \text{right}_i$

What if in the algorithm, we divide by 0?

This would only be the case if $h_i = 0$ and $h_{i-1} = 0$, so three points are superimposed.

In this case, the whole row of the matrix would be zero, which would have no influence, because the length of the segment to be drawn would also be zero. This problem is avoided by making the zero diagonals equal to 1.

Let us study the situation when the slopes at the edges are fixed.

We assume given p_1 and p_n. This will influence the values of M_1 and M_n.

We have seen that for i = 1.. nbPts - 1

$s'_i(t_i)=\frac{h_i}{6}\cdot(-2M_i-M_{i+1})+\frac{y_{i+1}-y_i}{h_i}$ and $h_{i-1}\cdot M_{i-1}+2(h_{i-1}+h_i)\cdot M_i+h_i\cdot M_{i+1}=6\cdot(\delta_i-\delta_{i-1})$.

So $p_1=s'_1(t_1)=\frac{h_1}{6}\cdot(-2M_1-M_2)+\frac{y_2-y_1}{h_1}$ => $M_1=3\cdot\left(\frac{y_2-y_1}{h_1^2}-\frac{p_1}{h_1}\right)-\frac{M_2}{2}$

We must also satisfy:

$h_1\cdot M_1+2(h_1+h_2)\cdot M_2+h_2\cdot M_3=6\cdot(\delta_2-\delta_1)$

Combining, we get: $\left(\frac{3}{2}h_1+2h_2\right)\cdot M_2+h_2\cdot M_3=6\cdot\delta_2-9\cdot\delta_1+3\cdot p_1$.

This changes the value of the first number on the diagonal, as well as the value of r_2.

If we have chosen the *"natural" situation where $M_1 = 0$*, then: $p_1=\frac{y_2-y_1}{h_1}-\frac{h_1}{6}\cdot M_2$ ($y_0=y_1-p_1$)

In any case, we have: $p_1=\frac{y_2-y_1}{h_1}-\frac{h_1}{6}\cdot(2M_1+M_2)$ ($V_0=V_1-p_1$)

At the end of the curve:

We have seen that: $s'_{i-1}(t_i)=\frac{h_{i-1}}{6}\cdot(2M_i+M_{i-1})+\frac{y_i-y_{i-1}}{h_{i-1}}$

So $p_n=s'_{n-1}(t_n)=\frac{h_{n-1}}{6}\cdot(2M_n+M_{n-1})+\frac{y_n-y_{n-1}}{h_{n-1}}$ => $M_n=3\cdot\left(\frac{p_n}{h_{n-1}}-\frac{y_n-y_{n-1}}{h_{n-1}^2}\right)-\frac{M_{n-1}}{2}$

We must also satisfy:

$h_{n-2}\cdot M_{n-2}+2(h_{n-2}+h_{n-1})\cdot M_{n-1}+h_{n-1}\cdot M_n=6\cdot(\delta_{n-1}-\delta_{n-2})$

Combining, we get: $\left(\frac{3}{2}h_{n-1}+2h_{n-2}\right)\cdot M_{n-1}+h_{n-2}\cdot M_{n-2}=9\cdot\delta_{n-1}-6\cdot\delta_{n-2}-3\cdot p_n$.

This changes the value of the last number on the diagonal, as well as the value of r_{n-1}.

If we have chosen the *"natural" situation where $M_n = 0$*, then: $p_n=\frac{y_n-y_{n-1}}{h_{n-1}}+\frac{h_{n-1}}{6}\cdot M_{n-1}$ ($y_{n+1}=y_n+p_n$)

In any case, we have: $p_n=\frac{y_n-y_{n-1}}{h_{n-1}}+\frac{h_{n-1}}{6}\cdot(2M_n+M_{n-1})$ ($V_{n+1}=V_n+p_n$)

Another possibility is to increase the dimension of the matrix to be solved by two.

In the case where the slopes are given at the edges, here is the matrix to solve:

$$\begin{pmatrix} 2\cdot h_1 & h_1 & 0 & 0 & 0 & 0 & 0 \\ h_1 & 2\cdot(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 2\cdot(h_2+h_3) & h_3 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 2\cdot(h_3+h_4) & h_4 & 0 & 0 \\ 0 & 0 & 0 & h_4 & 2\cdot(h_4+h_5) & h_5 & 0 \\ 0 & 0 & 0 & 0 & h_{n-2} & 2\cdot(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & 0 & h_{n-1} & 2\cdot h_{n-1} \end{pmatrix} \circ \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_{n-1} \\ r_n \end{pmatrix}$$

$n = 7$ in the matrix example above

$h_i = t_{i+1} - t_i \quad \delta_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = \frac{y_{i+1} - y_i}{h_i}$, for i = 1.. nbPts - 1

$\delta_i = 0$ if the numerator is zero, even if the denominator is also zero. It's arbitrary, but practical.

$r_i = 6\cdot(\delta_i - \delta_{i-1}) \quad i$ =2..n - 1

$r_1 = 6\cdot(\delta_1 - p_1); r_n = 6\cdot(p_n - \delta_{n-1})$

Resolution :

$h_0 = 0; h_n = 0$

$diag_i = 2\cdot(h_{i-1} + h_i)$; If $diag_i = 0$, then $diag_i = 1$ $\;i$ = 1.. n The first line is i =1

$lft_i = h_{i-1} \quad i = 1..n$

for i =2 to n do $diag_i = diag_i - \frac{lft_i}{diag_{i-1}} \cdot lft_i$ and $r_i = r_i - \frac{lft_i}{diag_{i-1}} \cdot r_{i-1}$ we have: $lft_i = \text{right}_{i-1}$

$M_n = \frac{r_n}{diag_n}$

for $i = n$ -1 downto 1 do $M_i = \frac{r_i - lft_{i+1}\cdot M_{i+1}}{diag_i}$ we have: $lft_{i+1} = \text{right}_i$

There are n + 2 points; n = nb_points - 2.

Study of the case where an interval has zero length.
Take the case where $h_4 = 0$, in this case the matrix becomes:

$$\begin{pmatrix} 2\cdot(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot(h_2+h_3) & h_3 & 0 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot h_5 & h_5 & 0 & 0 \\ 0 & 0 & 0 & h_5 & 2\cdot(h_5+h_6) & h_6 & 0 \\ 0 & 0 & 0 & 0 & h_{n-3} & 2\cdot(h_{n-3}+h_{n-2}) & h_{n-2} \\ 0 & 0 & 0 & 0 & 0 & h_{n-2} & 2\cdot(h_{n-2}+h_{n-1}) \end{pmatrix} \circ \begin{pmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_{n-1} \end{pmatrix} = \begin{pmatrix} r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_{n-1} \end{pmatrix}$$

$n = 9$ in the matrix example above

$h_i = t_{i+1} - t_i \quad \delta_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = \frac{y_{i+1} - y_i}{h_i}$, for i = 1.. nbPts - 1

$\delta_i = 0$ if the numerator is zero, even if the denominator is also zero. It's arbitrary, but practical.

$r_i = 6\cdot(\delta_i - \delta_{i-1})$ i =2..n - 1

The system splits into two independent systems.

$\delta_4 = \frac{y_5 - y_4}{h_4} = \frac{0}{0}$ is indeterminate.

So r_4 and r_5 are also indeterminate.
It would be practical and natural to have $M_4 = M_5 = 0$, so that the spline breaks down into two natural splines.
We would have :

$h_2\cdot M_2 + 2\cdot(h_2+h_3)\cdot M_3 = r_3$. The equation: $h_3\cdot M_3 + 2h_3\cdot M_4 = r_4$ would be replaced by $M_4 = 0$.

$2\cdot(h_5+h_6)\cdot M_6 + h_6\cdot M_7 = r_6$. The equation: $2h_5\cdot M_5 + h_5\cdot M_6 = r_5$ would be replaced by $M_5 = 0$.

Resolution :

$diag_i = 2\cdot(h_{i-1}+h_i)$; If $diag_i = 0$, then $diag_i = 1$ i = 2.. n -1 The first line is i =2

$lft_i = h_{i-1}$ i = 1.. n -1

$lft_4 = lft_5 = lft_6 = 0$ and $r_4 = r_5 = 0$ add to the algorithm.

for i =3 to n -1 do $diag_i = diag_i - \frac{lft_i}{diag_{i-1}}\cdot lft_i$ and $r_i = r_i - \frac{lft_i}{diag_{i-1}}\cdot r_{i-1}$ we have: $lft_i = \text{right}_{i-1}$

$M_{n-1} = \frac{r_{n-1}}{diag_{n-1}}$

for i = n -2 downto 2 do $M_i = \frac{r_i - lft_{i+1}\cdot M_{i+1}}{diag_i}$ we have: $lft_{i+1} = \text{right}_i$

We automatically have the two desired equalities: $M_4 = M_5 = 0$.

Take the case where we do not want continuity of the derivatives at $t = t_4$

This would make it possible to have a break in the slope at t_4.

The strikethrough line in the system below should be eliminated, leaving a degree of freedom.

M_4 of segment s_4 need not be equal to M_4 of segment s_5.

It would be natural to set $M_4 = 0$, so that the spline breaks down into two natural splines.

$$\begin{pmatrix} 2\cdot(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot(h_2+h_3) & h_3 & 0 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot(h_3+h_4) & h_4 & 0 & 0 & 0 \\ 0 & 0 & h_4 & 2\cdot(h_4+h_5) & h_5 & 0 & 0 \\ 0 & 0 & 0 & h_5 & 2\cdot(h_5+h_6) & h_6 & 0 \\ 0 & 0 & 0 & 0 & h_{n-3} & 2\cdot(h_{n-3}+h_{n-2}) & h_{n-2} \\ 0 & 0 & 0 & 0 & 0 & h_{n-2} & 2\cdot(h_{n-2}+h_{n-1}) \end{pmatrix} \circ \begin{pmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_{n-1} \end{pmatrix} = \begin{pmatrix} r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_{n-1} \end{pmatrix}$$

$n = 9$ in the matrix example above

$h_i = t_{i+1} - t_i \quad \delta_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = \frac{y_{i+1} - y_i}{h_i}$, for $i = 1..$ nbPts - 1

$\delta_i = 0$ if the numerator is zero, even if the denominator is also zero. It's arbitrary, but practical.

$r_i = 6\cdot(\delta_i - \delta_{i-1}) \quad i = 2..n-1$

Resolution :

We leave the line crossed out, $i = 4$, which will automatically give: $M_4 = 0$.

$diag_i = 2\cdot(h_{i-1}+h_i)$; If $diag_i = 0$, then $diag_i = 1 \quad i = 2..\ n - 1$ The first line is $i = 2$

$lft_i = h_{i-1} \quad i = 1..\ n - 1$

$lft_4 = lft_5 = 0$ and $r_4 = 0$ add to the algorithm.

for $i = 3$ to $n - 1$ do $diag_i = diag_i - \frac{lft_i}{diag_{i-1}} \cdot lft_i$ and $r_i = r_i - \frac{lft_i}{diag_{i-1}} \cdot r_{i-1}$ we have: $lft_i = \text{right}_{i-1}$

$M_{n-1} = \frac{r_{n-1}}{diag_{n-1}}$

for $i = n - 2$ downto 2 do $M_i = \frac{r_i - lft_{i+1}\cdot M_{i+1}}{diag_i}$ we have: $lft_{i+1} = \text{right}_i$

We automatically have the desired equality : $M_4 = 0$.

Case of periodic spline, corresponding to closed curves.
In this case, the second derivative of the curve at the “starting point” in M_1 and
the second derivative of the curve at the “end point” in M_{n+10} are approximated .
We add 10 points periodically, we eliminate the first 5 and last 5 segments, which gives us the periodic curve, because beyond 5 points, the segment is no longer influenced by the points.

Approximation of the second derivatives at the start and at the finish:
For a curve passing through points $\vec{v}_i=(x_i; y_i)$ it is different than for a spline function.
Here we deal with the case of a curve passing through points.
Given: $\vec{v}_i=(x_i; y_i)$ for i = 1.. nbPts
We add: $\vec{v}_0=\vec{v}_n$ and we can add: $\vec{v}_{n+i}=\vec{v}_i$.

h_i= distance between $\vec{v}_i$ and $\vec{v}_{i+1}=\|\vec{v}_{i+1}-\vec{v}_i\|$ $\delta_i=\frac{y_{i+1}-y_i}{h_i}$= slope , for i = 0.. nbPts +10

$M_1=2\cdot\frac{\delta_1-\delta_0}{h_1+h_0}$ and $M_m=2\cdot\frac{\delta_m-\delta_{m-1}}{h_m+h_{m-1}}$ n = nbPts and $m = n + 10$

System of equations to be solved to obtain the value of M_i $i = 2..n+10-1$
n = number of points = nbPts

$$\begin{pmatrix} 2\cdot(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 \\ h_2 & 2\cdot(h_2+h_3) & h_3 & 0 & 0 & 0 \\ 0 & h_3 & 2\cdot(h_3+h_4) & h_4 & 0 & 0 \\ 0 & 0 & h_4 & 2\cdot(h_4+h_5) & h_5 & 0 \\ 0 & 0 & 0 & h_{m-3} & 2\cdot(h_{m-3}+h_{m-2}) & h_{m-2} \\ 0 & 0 & 0 & 0 & h_{m-2} & 2\cdot(h_{m-2}+h_{m-1}) \end{pmatrix}\circ\begin{pmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_{m-2} \\ M_{m-1} \end{pmatrix}=\begin{pmatrix} r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_{m-2} \\ r_{m-1} \end{pmatrix}$$

m = 8 in the matrix example above

$r_i=6\cdot(\delta_i-\delta_{i-1})$ $i = 3..\ m-2$
$r_2=6\cdot(\delta_2-\delta_1)-h_1\cdot M_1$ and $r_{m-1}=6\cdot(\delta_{m-1}-\delta_{m-2})-h_{m-1}\cdot M_m$

Resolution :
$diag_i=2\cdot(h_{i-1}+h_i)$; If $diag_i=0$, then $diag_i=1$ $i = 2..\ m-1$ The first line is i =2
$lft_i=h_{i-1}$ $i = 1..\ m-1$

for i =3 to m -1 do $diag_i=diag_i-\frac{lft_i}{diag_{i-1}}\cdot lft_i$ and $r_i=r_i-\frac{lft_i}{diag_{i-1}}\cdot r_{i-1}$ we have: $lft_i=\text{right}_{i-1}$

$M_{m-1}=\frac{r_{m-1}}{diag_{m-1}}$

for $i = m$ -2 downto 2 do $M_i=\frac{r_i-lft_{i+1}\cdot M_{i+1}}{diag_i}$ we have: $lft_{i+1}=\text{right}_i$

We only keep the values from M_5 to M_{m-5}. Note $(M_{m-5}=M_{n+5})$
to define segments from 1 to n .

Case of periodic spline, corresponding to closed curves, but with break.
In the case of a closed curve, with a break at a point, that is to say with a point having a discontinuity of the tangent, this corresponds to a non-periodic spline, with a beginning and an end superimposed on the point having the break.
For the calculations, you just have to start the matrix at the point having the break.

The other way is to add the following conditions to each breakout k :
$lft_k = lft_{k+1} = 0$ and $r_k = 0$ to add in the algorithm, before the elimination of Gauss and to add points, as described previously.
It's easier.

Links with other spline curves, described in the video “The continuity of Splines. »

See: https://www.youtube.com/watch?v=jvPPXbo87ds
by Freya Holmér see: https://www.youtube.com/@Acegikmo

Excellent video which summarizes different approximations by splines.
Unfortunately, the spline I described earlier is not part of the list of splines described in this video.

Bezier:

$$P(t)=\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\circ\begin{pmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}P_0\\ P_1\\ P_2\\ P_3\end{pmatrix}$$

P_1 et P_2 sont des points de contrôle.
$P(0) = P_0$
$P(1) = P_3$
$P'(0) = -3\ P_0 + 3\ P_1$
$P'(1) = -3\ P_2 + 3\ P_3$

Hermit:

$$P(t)=\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\circ\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3 & -2 & 3 & -1\\ 2 & 1 & -2 & 1\end{pmatrix}\circ\begin{pmatrix}P_0\\ V_0\\ P_1\\ V_1\end{pmatrix}$$

$P(0) = P_0$
$P(1) = P_1$
$P'(0) = V_0$
$P'(1) = V_1$

Catmull-Rom:

$$P(t)=\frac{1}{2}\cdot\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\circ\begin{pmatrix}0 & 2 & 0 & 0\\ -1 & 0 & 1 & 0\\ 2 & -5 & 4 & -1\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}P_0\\ P_1\\ P_2\\ P_3\end{pmatrix}$$

$P(0) = P_1$
$P(1) = P_2$
$P'(0) = -0.5\ P_0 + 0.5\ P_2$
$P'(1) = -0.5\ P_1 + 0.5\ P_3$

B-Spline:

$$P(t)=\frac{1}{6}\cdot\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\circ\begin{pmatrix}1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}P_0\\ P_1\\ P_2\\ P_3\end{pmatrix}$$

$P(0) = (P_0 + 4\ P_1 + P_2) / 6$
$P(1) = (P_1 + 4P_2 + P_3) / 6$
$P'(0) = -0.5\ P_0 + 0.5\ P_2$
$P'(1) = -0.5\ P_1 + 0.5\ P_3$

Parts of the video:
22' - 23', improvement, to obtain the spline described in the previous pages.
38' Geometric continuity
43 Hermit Bezier
46' - 47' Cardinal spline
48' Catmull-Rom spline
54' Matrix of B-spline
57' Summary of the 4 splines above.

More precise definition of a B-spline and links with Math-splines

After having more precisely defined the B-spline curve obtained by control points, we will see how to pass from the control points of a B-spline to the control points of a Math -spline and vice versa.
Both curves are the same, just the way to control them using various points differ.
The advantage of a B-spline is that moving a control point only changes the 4 adjacent segments. Its disadvantage is that it does not pass through checkpoints.
The advantage of a Math-spline is to pass through all the control points, except the first and the last which we will define later. Its disadvantage is that it theoretically modifies all the segments of the curve, although practically it only modifies the 8 adjacent segments.

To avoid confusion with the letters used previously, here are notation conventions:
The control points of the B-spline will be denoted: U_i, for $i = 0 .. n+1$.
The Math-spline control points will be denoted: V_i, for $i = 0 .. n+1$.
The U_i and the V_i are points or vectors of dimension 2 for a plane and of higher dimension for a larger space. (3 for our space and more for a mathematician.)

The spline curve will be denoted $s(t)$, for $t = t_1$ to t_n, satisfying: $s(t_i)=V_i$, for $i = 1 .. n$.
It is defined by
$s(t)=s_i(t)=s_i(t_i+\tau\cdot h_i)$ to $t\in[t_i..t_{i+1}]$ $\tau\in[0..1]$

$s_i(t)$ is piecewise cubic.

Simplification for a first approach.
$t_i=i$ and so $h_i=1$ for all i .
Let us look for a matrix A making the spline curve twice continuously differentiable, to find the matrix linked to a B-spline.

$$s_i(t_i+\tau)=\begin{pmatrix}1 & \tau & \tau^2 & \tau^3\end{pmatrix}\circ\begin{pmatrix}a_{1;1} & a_{1;2} & a_{1;3} & a_{1;4}\\ a_{2;1} & a_{2;2} & a_{2;3} & a_{2;4}\\ a_{3;1} & a_{3;2} & a_{3;3} & a_{3;4}\\ a_{4;1} & a_{4;2} & a_{4;3} & a_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

$$s'_i(t_i+\tau)=\begin{pmatrix}0 & 1 & 2\tau & 3\tau^2\end{pmatrix}\circ\begin{pmatrix}a_{1;1} & a_{1;2} & a_{1;3} & a_{1;4}\\ a_{2;1} & a_{2;2} & a_{2;3} & a_{2;4}\\ a_{3;1} & a_{3;2} & a_{3;3} & a_{3;4}\\ a_{4;1} & a_{4;2} & a_{4;3} & a_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

$$s''_i(t_i+\tau)=\begin{pmatrix}0 & 0 & 2 & 6\tau\end{pmatrix}\circ\begin{pmatrix}a_{1;1} & a_{1;2} & a_{1;3} & a_{1;4}\\ a_{2;1} & a_{2;2} & a_{2;3} & a_{2;4}\\ a_{3;1} & a_{3;2} & a_{3;3} & a_{3;4}\\ a_{4;1} & a_{4;2} & a_{4;3} & a_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

For any U_i, the conditions to be met are:
$s_i(t_{i+1})=s_{i+1}(t_{i+1})$
$s_i(t_{i+1})=s_{i+1}(t_{i+1})$
$s_i''(t_{i+1})=s_{i+1}''(t_{i+1})$

You can skip the calculations on the next page, to see only the result.

$s_i(t_{i+1})=s_{i+1}(t_{i+1})$ imposes the following condition, which breaks down into 5 conditions:
$(a_{1;1}+a_{2;1}+a_{3;1}+a_{4;1})\cdot U_{i-1}+$
$(a_{1;2}+a_{2;2}+a_{3;2}+a_{4;2})\cdot U_i+$
$(a_{1;3}+a_{2;3}+a_{3;3}+a_{4;3})\cdot U_{i+1}+$
$(a_{1;4}+a_{2;4}+a_{3;4}+a_{4;4})\cdot U_{i+2}=$
$a_{1;1}\cdot U_i+a_{1;2}\cdot U_{i+1}+a_{1;3}\cdot U_{i+2}+a_{1;4}\cdot U_{i+3}$
Since the equality must be true for all U_i, it gives 5 equations.

$s'_i(t_{i+1})=s'_{i+1}(t_{i+1})$ imposes the following condition, which breaks down into 5 conditions:
$(a_{2;1}+2a_{3;1}+3a_{4;1})\cdot U_{i-1}+$
$(a_{2;2}+2a_{3;2}+3a_{4;2})\cdot U_i+$
$(a_{2;3}+2a_{3;3}+3a_{4;3})\cdot U_{i+1}+$
$(a_{2;4}+2a_{3;4}+3a_{4;4})\cdot U_{i+2}=$
$a_{2;1}\cdot U_i+a_{2;2}\cdot U_{i+1}+a_{2;3}\cdot U_{i+2}+a_{2;4}\cdot U_{i+3}$
Since the equality must be true for all U_i, it gives 5 equations.

$s''_i(t_{i+1})=s''_{i+1}(t_{i+1})$ imposes the following condition, which breaks down into 5 conditions:
$(2a_{3;1}+6a_{4;1})\cdot U_{i-1}+$
$(2a_{3;2}+6a_{4;2})\cdot U_i+$
$(2a_{3;3}+6a_{4;3})\cdot U_{i+1}+$
$(2a_{3;4}+6a_{4;4})\cdot U_{i+2}=$
$2a_{3;1}\cdot U_i+2a_{3;2}\cdot U_{i+1}+2a_{3;3}\cdot U_{i+2}+2a_{3;4}\cdot U_{i+3}$
Since the equality must be true for all U_i, it gives 5 equations.

We obtain 15 equations quite easy to solve, with a free unknown.
Concerning the U_{i+3} we obtain: $a_{1;4}=0$; $a_{2;4}=0$;$a_{3;4}=0$

Concerning the U_{i+2} we obtain:
$2a_{3;4}+6a_{4;4}=2a_{3;3}$, So $a_{3;3}=3a_{4;4}$
$a_{2;4}+2a_{3;4}+3a_{4;4}=a_{2;3}$, So $a_{2;3}=3a_{4;4}$
$a_{1;4}+a_{2;4}+a_{3;4}+a_{4;4}=a_{1;3}$, So $a_{1;3}=a_{4;4}$

Concerning the U_{i+1} we obtain:
$2a_{3;3}+6a_{4;3}=2a_{3;2}$, So $a_{3;2}=3a_{4;4}+3a_{4;3}$
$a_{2;3}+2a_{3;3}+3a_{4;3}=a_{2;2}$, So $a_{2;2}=9a_{4;4}+3a_{4;3}$
$a_{1;3}+a_{2;3}+a_{3;3}+a_{4;3}=a_{1;2}$, So $a_{1;2}=7a_{4;4}+a_{4;3}$

Concerning the U_i we obtain:
$2a_{3;2}+6a_{4;2}=2a_{3;1}$, So $a_{3;1}=3a_{4;4}+3a_{4;3}+3a_{4;2}$
$a_{2;2}+2a_{3;2}+3a_{4;2}=a_{2;1}$, So $a_{2;1}=15a_{4;4}+9a_{4;3}+3a_{4;2}$
$a_{1;2}+a_{2;2}+a_{3;2}+a_{4;2}=a_{1;1}$, So $a_{1;1}=19a_{4;4}+7a_{4;3}+a_{4;2}$

Concerning the U_{i-1} we obtain:
$2a_{3;1}+6a_{4;1}=0$, So $a_{3;1}=-3a_{4;1}$
$a_{2;1}+2a_{3;1}+3a_{4;1}=0$, So $a_{2;1}=3a_{4;1}$
$a_{1;1}+a_{2;1}+a_{3;1}+a_{4;1}=0$, So $a_{1;1}=-a_{4;1}$

By solving 3 equations, we get: $a_{4;3}=-3a_{4;4}$ and $a_{4;2}=3a_{4;4}$ and $a_{4;1}=-a_{4;4}$
Which constraint must still be satisfied to fix $a_{4;4}$?

We obtain :

$$s_i(t_i+\tau)=a_{4;4}\cdot\begin{pmatrix}1 & \tau & \tau^2 & \tau^3\end{pmatrix}\circ\begin{pmatrix}1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix},\tau\in[0..1]$$

We find the matrix of the B-spline curve, with a free parameter, which corresponds to subjecting the whole curve to a homothety.
For that $s_i(t_i)=a_{4;4}\cdot(U_{i-1}+4U_i+U_{i+1})$corresponds to a weighted average of the U_i, we choose the value of $a_{4;4}=\frac{1}{6}$. Another choice would make the curve dependent on the choice of plane origin. It is necessary that if we translate all the U_i in the same way, that we calculate the B-spline curve, then that we translate it in the other direction, we obtain the same curve independently of the translation.

Thus we obtain exactly the characteristic matrix of a B-spline.

$$s_i(t_i+\tau)=\frac{1}{6}\cdot\begin{pmatrix}1 & \tau & \tau^2 & \tau^3\end{pmatrix}\circ\begin{pmatrix}1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

In appendix II, I recheck that the curve thus obtained is indeed twice continuously differentiable.

A first relation between the control points of the B-spline and those of the Math-spline:
$s(t_i)=s_i(t_i)=V_i=\frac{1}{6}\cdot(U_{i-1}+4U_i+U_{i+1})$, for $i = 1..n$.

The two control points V_0 and $V_{n+1\text{ remain}}$.
The Math-spline has two degrees of freedom corresponding to the slope of the curve at the start and the slope of the curve at the finish.

$$\begin{aligned}s(t_i+\tau)=\frac{1}{6}\cdot\big[&U_{i-1}+4U_i+U_{i+1}+\\ &\tau\cdot(-3U_{i-1}+3U_{i+1})+\\ &\tau^2\cdot(3U_{i-1}-6U_i+3U_{i+1})+\\ &\tau^3\cdot(-U_{i-1}+3U_i-3U_{i+1}+U_{i+2})\big]\end{aligned}$$

<u>Derived from s(t).</u>

$$\begin{aligned}s'_i(t_i+\tau)=\frac{1}{6}\cdot\big[&-3U_{i-1}+3U_{i+1}+\\ &\tau\cdot(6U_{i-1}-12U_i+6U_{i+1})+\\ &\tau^2\cdot(-3U_{i-1}+9U_i-9U_{i+1}+3U_{i+2})\big]\end{aligned}$$

$$s'_i(t_i+\tau)=\frac{1}{6}\cdot\begin{pmatrix}0 & 1 & 2\tau & 3\tau^2\end{pmatrix}\circ\begin{pmatrix}1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}$$

<u>Let's calculate the second derivative of s(t).</u>

$$\begin{aligned}s''_i(t_i+\tau)=&U_{i-1}-2U_i+U_{i+1}+\\ &\tau\cdot(-U_{i-1}+3U_i-3U_{i+1}+U_{i+2})\big]\end{aligned}$$

Determination of the two control points V_0 and V_{n+1} of the Math-spline to make the two curves coincide.

Slope at start = $p_1 = s'(t_1) = s'_1(t_1) = \frac{1}{2} \cdot U_2 - \frac{1}{2} \cdot U_0$.

Slope on arrival = $p_n = s'(t_n) = s'_{n-1}(t_n) = \frac{1}{2} \cdot U_{n+1} - \frac{1}{2} \cdot U_{n-1}$.

For the Math-spline, these two slopes are free to choose.
It remains to define a link between these two slopes and the two values V_0 and V_{n+1}.
We want the following conditions to be fulfilled:

1) $p_1 = \frac{1}{2} \cdot U_2 - \frac{1}{2} \cdot U_0$

2) $V_0 = \alpha \cdot U_0 + \beta \cdot U_1 + \gamma \cdot U_2$, with $\alpha ; \beta ; \gamma$ free

3) $V_1 = \frac{1}{6} \cdot U_0 + \frac{4}{6} \cdot U_1 + \frac{1}{6} \cdot U_2$

4) $p_1 = \lambda \cdot V_1 - \mu \cdot V_0$, with $\lambda ; \mu$ free

We therefore want to express V_0 as a function of U_0, U_1 and U_2, which makes it possible to determine p_1 which satisfies condition 1), which describes the derivative at the start of the curve.

From these 4 equalities, let p_1, V_0 and V_1 disappear.
1) = 4) and substitute 2) and 3), to obtain:

$$\frac{1}{2} \cdot U_2 - \frac{1}{2} \cdot U_0 = \lambda \cdot \frac{1}{6} \cdot U_0 + \lambda \cdot \frac{4}{6} \cdot U_1 + \lambda \cdot \frac{1}{6} \cdot U_2 - \mu \cdot \alpha \cdot U_0 - \mu \cdot \beta \cdot U_1 - \mu \cdot \gamma \cdot U_2$$

By rearranging the terms and highlighting the U_i:

$$U_0 \cdot \left(\frac{1}{2} + \lambda \cdot \frac{1}{6} - \mu \cdot \alpha \right) + U_1 \cdot \left(\lambda \cdot \frac{4}{6} - \mu \cdot \beta \right) + U_2 \cdot \left(\lambda \cdot \frac{1}{6} - \mu \cdot \gamma - \frac{1}{2} \right) = 0$$

We want the equality to be true regardless of the values of U_i, so we must:

$\frac{1}{2} + \lambda \cdot \frac{1}{6} - \mu \cdot \alpha = 0$ and $\qquad \alpha = \frac{1}{2 \cdot \mu} + \frac{\lambda}{6 \mu}$

$\lambda \cdot \frac{4}{6} - \mu \cdot \beta = 0$ and $\qquad \beta = \frac{4 \lambda}{6 \mu}$

$\lambda \cdot \frac{1}{6} - \mu \cdot \gamma - \frac{1}{2} = 0 \qquad \gamma = \frac{\lambda}{6 \mu} - \frac{1}{2 \cdot \mu}$

A natural choice is: $\lambda = \mu = 1$. With this choice, we get: $p_1 = V_1 - V_0$ and
$\alpha = \frac{4}{6}$; $\beta = \frac{4}{6}$; $\gamma = -\frac{2}{6}$.

So : $V_0 = \frac{4}{6} \cdot U_0 + \frac{4}{6} \cdot U_1 - \frac{2}{6} \cdot U_2$

Parenthesis, not accepted.
A pleasant choice to obtain a tri-diagonal matrix for passing from U_i to V_i would be to have
$\gamma = \frac{\lambda}{6 \mu} - \frac{1}{2 \cdot \mu} = 0$, therefore $\lambda = 3 \cdot \mu = 3$ and $\mu = 1$
$\alpha = \frac{3}{2}$ and $\beta = 2$, $V_0 = \frac{3}{2} \cdot U_0 + 2 \cdot U_1$, $p_1 = 3 \cdot V_1 - V_0$. For p_1 this is not natural. FRO.

Let's do similar calculations to determine V_{n+1}.

We want the following conditions to be fulfilled:

1) $p_n = -\frac{1}{2} \cdot U_{n-1} + \frac{1}{2} \cdot U_{n+1}$

2) $V_{n+1} = \alpha \cdot U_{n-1} + \beta \cdot U_n + \gamma \cdot U_{n+1}$, with $\alpha ; \beta ; \gamma$ free

3) $V_n = \frac{1}{6} \cdot U_{n-1} + \frac{4}{6} \cdot U_n + \frac{1}{6} \cdot U_{n+1}$

4) $p_n = \mu \cdot V_{n+1} - \lambda \cdot V_n$, with $\lambda ; \mu$ free

We therefore want to express V_{n+1} as a function of U_{n-1}, U_n and U_{n+1}, which makes it possible to determine p_n which satisfies condition 1), which describes the derivative at the end of the curve.

From these 4 equalities, let p_n, V_n and V_{n+1} disappear.
1) = 4) and substitute 2) and 3), to obtain:

$$\frac{1}{2} \cdot U_{n+1} - \frac{1}{2} \cdot U_{n-1} = \mu \cdot \alpha \cdot U_{n-1} + \mu \cdot \beta \cdot U_n + \mu \cdot \gamma \cdot U_{n+1} - \lambda \cdot \frac{1}{6} \cdot U_{n-1} - \lambda \cdot \frac{4}{6} \cdot U_n - \lambda \cdot \frac{1}{6} \cdot U_{n+1}$$

By rearranging the terms and highlighting the U_i:

$$U_{n-1} \cdot \left(\frac{1}{2} - \lambda \cdot \frac{1}{6} + \mu \cdot \alpha \right) + U_n \cdot \left(\mu \cdot \beta - \lambda \cdot \frac{4}{6} \right) + U_{n+1} \cdot \left(\mu \cdot \gamma - \lambda \cdot \frac{1}{6} - \frac{1}{2} \right) = 0$$

We want the equality to be true regardless of the values of U_i, so we must:

$\frac{1}{2} - \lambda \cdot \frac{1}{6} + \mu \cdot \alpha = 0$ and $\qquad \alpha = \frac{\lambda}{6\mu} - \frac{1}{2 \cdot \mu}$

$\mu \cdot \beta - \lambda \cdot \frac{4}{6} = 0$ and $\qquad \beta = \frac{4\lambda}{6\mu}$

$\mu \cdot \gamma - \lambda \cdot \frac{1}{6} - \frac{1}{2} = 0$ $\qquad \gamma = \frac{\lambda}{6\mu} + \frac{1}{2 \cdot \mu}$

A natural choice is: $\lambda = \mu = 1$. With this choice, we get: $p_n = V_{n+1} - V_n$ and
$\alpha = -\frac{2}{6}$; $\beta = \frac{4}{6}$; $\gamma = \frac{4}{6}$.

So : $V_{n+1} = -\frac{2}{6} \cdot U_{n-1} + \frac{4}{6} \cdot U_n + \frac{4}{6} \cdot U_{n+1}$

To calculate the curve of the B-spline passing through the control points:

$$\begin{aligned} s_i(t_i + \tau) = \frac{1}{6} \cdot [& U_{i-1} + 4U_i + U_{i+1} + \\ & \tau \cdot (-3U_{i-1} + 3U_{i+1}) + \\ & \tau^2 \cdot (3U_{i-1} - 6U_i + 3U_{i+1}) + \\ & \tau^3 \cdot (-U_{i-1} + 3U_i - 3U_{i+1} + U_{i+2})] \quad \tau \in [0..1] \end{aligned}$$

Writing in matrix form of the transition from U_i to V_i.

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_n \\ V_{n+1} \end{pmatrix} = \frac{1}{6} \cdot \begin{pmatrix} 4 & 4 & -2 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & -2 & 4 & 4 \end{pmatrix} \circ \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_n \\ U_{n+1} \end{pmatrix}$$

Here, $n = 4$. There are 4 waypoints and 6 control points.

So the transition from the B-spline control points to the Math-spline control points is:

$V_0 = \frac{4}{6} \cdot U_0 + \frac{4}{6} \cdot U_1 - \frac{2}{6} \cdot U_2; V_{n+1} = -\frac{2}{6} \cdot U_{n-1} + \frac{4}{6} \cdot U_n + \frac{4}{6} \cdot U_{n+1}$

$V_i = \frac{1}{6} \cdot U_{i-1} + \frac{4}{6} \cdot U_i + \frac{1}{6} \cdot U_{i+1}$, for $i = 1.. n$

Passing from Math-spline control points to B-spline control points.

Going from Math-spline control points to B-spline control points requires solving the system of equations. Since the matrix is tri-diagonal with dominant diagonal, the calculation is quite fast. The V_i are given, we seek the U_i.

To get a tri-diagonal matrix, let's combine the first two rows and the last two.

$$\begin{pmatrix} 6 & 12 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 12 & 6 \end{pmatrix} \circ \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_n \\ U_{n+1} \end{pmatrix} = 6 \cdot \begin{pmatrix} V_0 + 2V_1 \\ V_1 \\ V_2 \\ V_3 \\ V_n \\ V_{n+1} + 2V_n \end{pmatrix}$$

Resolution :

$diag_0 = 6$; $diag_{n+1} = 6$;

$diag_i = 4$, for $i = 1.. n$

$q_0 = 6 \cdot (V_0 + 2 \cdot V_1)$, $q_{n+1} = 6 \cdot (V_{n+1} + 2 \cdot V_n)$, $q_i = 6 \cdot V_i$, for $i = 1.. n$

$diag_1 = diag_1 - \frac{1}{diag_0} \cdot 12$ and $q_1 = q_1 - \frac{1}{diag_0} \cdot q_0$

for i=2 to n do $diag_i = diag_i - \frac{1}{diag_{i-1}} \cdot 1$ and $q_i = q_i - \frac{1}{diag_{i-1}} \cdot q_{i-1}$

$diag_{n+1} = diag_{n+1} - \frac{12}{diag_n} \cdot 1$ and $q_{n+1} = q_{n+1} - \frac{12}{diag_n} \cdot q_n$

$U_{n+1} = \frac{q_{n+1}}{diag_{n+1}}$

for $i = n$ downto 1 do $U_i = \frac{q_i - 1 \cdot U_{i+1}}{diag_i}$ There are $n + 2$ points; n = nb_points - 2.

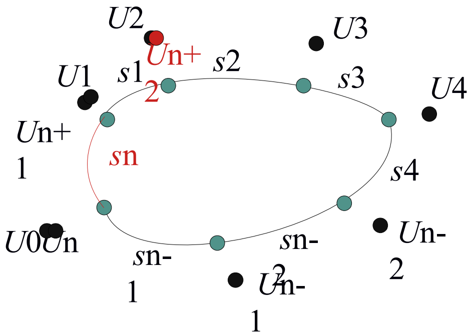
$U_0 = \frac{q_0 - 12 \cdot U_1}{diag_0}$

Out of curiosity, let's look at what has become of the system after triangulation by the Gaussian method described in the resolution above.

$$\begin{pmatrix} 6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.7142857 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.7307692 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.7319588 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7320442 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.7320503 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.7846093 \end{pmatrix} \circ \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_n \\ U_{n+1} \end{pmatrix} =$$

$$6 * \begin{pmatrix} V_0+2V_1 \\ 0.666666 \cdot V_1 - 0.166666 \cdot V_0 \\ V_2 - 0.333333 \cdot V_1 + 0.083333 \cdot V_0 \\ V_3 - 0.285714 \cdot V_2 + 0.095238 \cdot V_1 - 0.023810 \cdot V_0 \\ V_4 - 0.269231 \cdot V_3 + 0.076923 \cdot V_2 - 0.025641 \cdot V_1 + 0.006410 \cdot V_0 \\ V_5 - 0.268041 \cdot V_4 + 0.072165 \cdot V_3 - 0.020619 \cdot V_2 + 0.006873 \cdot V_1 - 0.001718 \cdot V_0 \\ V_6 - 0.267955 \cdot V_5 + 0.071823 \cdot V_4 - 0.019337 \cdot V_3 + 0.005525 \cdot V_2 - 0.001842 \cdot V_1 + 0.000460 \cdot V_0 \\ V_n - 0.267959 \cdot V_6 + 0.071799 \cdot V_5 - 0.019245 \cdot V_4 + 0.005181 \cdot V_3 - 0.001480 \cdot V_2 + 0.000493 \cdot V_1 - 0.000 \\ V_{n+1} - 0.267959 \cdot V_7 + 0.071799 \cdot V_6 - 0.019228 \cdot V_5 + 0.005157 \cdot V_4 - 0.001388 \cdot V_3 + 0.000397 \cdot V_2 - 0.000132 \cdot V \end{pmatrix}$$

We notice that very quickly the diagonal of the matrix converges towards $2+\sqrt{3}=3.732051$.
We also note that the V_i quickly no longer influence the remote U_i.



Closed B-splines.

In the case of the drawing, $n = 7$

The goal is to add a segment s_n, so that the curve is closed.
To do this, we will add a point U_{n+2} and the segment s_n associated with the points U_{n+2}, U_{n+1}, U_n and U_{n-1}, and allow the position of the points U_{n+1} to be modified, and U_0.
We want this segment to end on U_0, ie C^1 and C^2 at U_0.

The conditions of continuity, of continuity of the derivative and of the second derivative are satisfied between s_{n-1} and s_n if we have as usual:

$$s_n(t_n+\tau)=\frac{1}{6}\cdot\Big[U_{n-1}+4U_n+U_{n+1}+ \\ \tau\cdot(-3U_{n-1}+3U_{n+1})+ \\ \tau^2\cdot(3U_{n-1}-6U_i+3U_{n+1})+ \\ \tau^3\cdot(-U_{n-1}+3U_n-3U_{n+1}+U_{n+2})\Big]$$

On the other hand, three new conditions must be satisfied to have the continuity of the curve, the derivative and the second derivative.
It is therefore necessary to satisfy:

$s_n(t_{n+1}) = s_1(t_1)$, ($t_{n+1} = t_n + 1$). So $\frac{1}{6}\cdot[U_n+4U_{n+1}+U_{n+2}]=\frac{1}{6}\cdot[U_0+4U_1+U_2]$ and

$s'_n(t_{n+1}) = s'_1(t_1)$. So $\frac{1}{6}\cdot[3U_{n+2}-3U_n]=\frac{1}{6}\cdot[3U_2-3U_0]$

$s''_n(t_{n+1}) = s''_1(t_1)$. So $\frac{1}{6}\cdot[U_{n+2}-2U_{n+1}+U_n]=\frac{1}{6}\cdot[U_2-2U_1+U_0]$

We check that they are satisfied if and only if:
$U_{n+2}=U_2$ and $U_{n+1}=U_1$ and $U_0=U_n$

In conclusion, to close a B-spline, it is necessary to add a point U_{n+2}, place it on point U_2 and place point U_0 on U_n and place point U_{n+1} on U_1.

The closed Math-spline linked to the points V_0 to V_{n+1} corresponding to the points U_0 to U_{n+1}, will give the same curve as the closed B-spline described above.
Note that the closed B-spline has one more point than the closed Math-spline!

The modification of the position of the points U_0 and U_{n+1} will modify the position of the points V_0, V_1, V_n and V_{n+1}, therefore the Math-spline curve. But it will not change the position of other points.
For a closed Math-spline, points V_0 and V_{n+1} are ignored.
These modifications of the points U_0 and U_{n+1} make it possible to close the curve while keeping it smooth, without disturbing the segments s_2 to s_{n-2}.

If a curve of a Math-spline is closed, then the corresponding B-spline will automatically have $U_{n+1}=U_1$ and $U_0=U_n$. Closing the B-spline will give the same curve as the Math-spline.

Study of the general case, where the transit times t_i are no longer regular.
Let us summarize the characteristics of B-splines of degree 3.
A B-spline (of degree 3) is characterized by the data of:
° control points of the B-spline will be denoted: U_i, for $i = 0 .. n+1$;
° of passage time t_i for $i = 1 .. n$ and
° of n -1 4x4 characteristic matrices which will be denoted A_i
such that if we define:

$$s_i(t)=\begin{pmatrix}1 & t-t_i & (t-t_i)^2 & (t-t_i)^3\end{pmatrix}\circ\begin{pmatrix}a_{i,1;1} & a_{i,1;2} & a_{i,1;3} & a_{i,1;4}\\ a_{i,2;1} & a_{i,2;2} & a_{i,2;3} & a_{i,2;4}\\ a_{i,3;1} & a_{i,3;2} & a_{i,3;3} & a_{i,3;4}\\ a_{i,4;1} & a_{i,4;2} & a_{i,4;3} & a_{i,4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

The curve (B-spline) defined by $s(t)=s_i(t)$ for $t \in [t_i .. t_{i+1}]$,
is twice continuously differentiable.

The U_i are points or vectors of dimension 2 for a plane and of higher dimension for a larger space. (3 for our space and more for a mathematician.)

To make the connection with the above: $s(t_i)=s_i(t_i)=V_i$, for $i = 1 .. n$.
$s_i(t)$ is therefore piecewise cubic.

The question is how to determine the coefficients $a_{i,j;k}$ above so that the curve s(t) defined above is always twice continuously differentiable, whatever the points U_i.

This time, the coefficients $a_{i,j;k}$ will depend on i .
In order not to weigh down the writing, the *i -dependency* will not always be explicit.

In the next few pages, there are a lot of calculations! They can be skipped, to see the result in the summary that follows these calculations.

For this, let's start from the definition of a B-spline from the Wikipedia page:
https://en.wikipedia.org/wiki/B-spline
Let us limit the case of B-spline of degree 3, let us transform the notation to adapt it to that used previously.

Given m +1 nodes $\tilde{t}_i$ with $\tilde{t}_0 \le \tilde{t}_1 \le .. \le \tilde{t}_m$
a B-spline curve of degree 3 is a parametric curve $s(t)$ defined as follows:

$$s(t)=\sum_{j=0}^{m-4} \tilde{b}_{j;3}(t)\cdot P_j,\ t \in [\tilde{t}_3, \tilde{t}_{m-3}]$$

where the P_i form a polygon called the control polygon.
The number of points composing this polygon is equal to m - 3.
The m -3 B-spline functions of degree k are defined by induction on the lower degree:

$$\tilde{b}_{j;0}(t)=\begin{cases}1 & si\,\tilde{t}_j \le t < \tilde{t}_{j+1}\\ 0 & sinon\end{cases}$$

$$\tilde{b}_{j;k}(t)=\frac{t-\tilde{t}_j}{\tilde{t}_{j+k}-\tilde{t}_j}\cdot\tilde{b}_{j;k-1}(t)+\frac{\tilde{t}_{j+k+1}-t}{\tilde{t}_{j+k+1}-\tilde{t}_{j+1}}\cdot\tilde{b}_{j+1;k-1}(t)$$

$$\tilde{b}_{j;k}(t)=\frac{t-\tilde{t}_j}{\tilde{h}_{j;k}}\cdot\tilde{b}_{j;k-1}(t)+\frac{\tilde{t}_{j+k+1}-t}{\tilde{h}_{j+1;k}}\cdot\tilde{b}_{j+1;k-1}(t),\ \text{with}\ \tilde{h}_{j;k}=\tilde{t}_{j+k}-\tilde{t}_j$$

$\tilde{b}_{j;k}(t)$ is zero if $t \notin [\tilde{t}_j, \tilde{t}_{j+k+1}]$

Changed notations to match the one used previously.
$U_j = P_j; t_{j-2} = \tilde{t}_j$
m - 3 = n + 2 = number of control points
$n = m$ -5; $m = n + 5$
$b_{j-2;k}(t) = \tilde{b}_{j;k}(t)$

So
$$s(t) = \sum_{j=0}^{n+1} b_{j-2;3}(t) \cdot U_j, t \in [t_1, t_n]$$

$b_{j-2;0}(t) = \tilde{b}_{j;0}(t) = \begin{cases} 1 & si\, t_{j-2} \leq t < t_{j-1} \\ 0 & sinon \end{cases}$, it remains to define the values of t_{-2}, t_{-1} and t_0 !?!

$b_{j;0}(t) = \tilde{b}_{j+2;0}(t) = \begin{cases} 1 & si\, t_j \leq t < t_{j+1} \\ 0 & sinon \end{cases}$

$b_{j;k}(t) = \frac{t - t_j}{h_{j;k}} \cdot b_{j;k-1}(t) + \frac{t_{j+k+1} - t}{h_{j+1;k}} \cdot b_{j+1;k-1}(t)$, with $h_{j;k} = t_{j+k} - t_j$, so $h_j = h_{j;1} = t_{j+1} - t_j$

$b_{j;k}(t)$ is zero if $t \notin [t_j, t_{j+k+1}]$

$b_{j;3}(t)$ is zero if $t \notin]t_j, t_{j+4}[$ also: $b_{j-4;3}(t)$ is zero if $t \notin]t_{j-4}, t_j[$.

$$s_i(t) = \sum_{j=0}^{n+1} b_{j-2;3}(t) \cdot U_j = \sum_{j=i-1}^{j+2} b_{j-2;3}(t) \cdot U_j, t \in [t_i, t_{i+1}]$$

Thus we can clearly see that $s_i(t)$ only depends on the points U_{i-1} to U_{i+2}.

Let's explain them $b_{j;k}(t)$.

$b_{j;1}(t) = \frac{t - t_j}{h_{j;1}} \cdot b_{j;0}(t) + \frac{t_{j+2} - t}{h_{j+1;1}} \cdot b_{j+1;0}(t) \quad b_{j+1;1}(t) = \frac{t - t_{j+1}}{h_{j+1;1}} \cdot b_{j+1;0}(t) + \frac{t_{j+3} - t}{h_{j+2;1}} \cdot b_{j+2;0}(t)$

$b_{j;2}(t) = \frac{t - t_j}{h_{j;2}} \cdot b_{j;1}(t) + \frac{t_{j+3} - t}{h_{j+1;2}} \cdot b_{j+1;1}(t)$

$$b_{j;2}(t) = \frac{t - t_j}{h_{j;2}} \cdot \left(\frac{t - t_j}{h_{j;1}} \cdot b_{j;0}(t) + \frac{t_{j+2} - t}{h_{j+1;1}} \cdot b_{j+1;0}(t) \right) + \frac{t_{j+3} - t}{h_{j+1;2}} \cdot \left(\frac{t - t_{j+1}}{h_{j+1;1}} \cdot b_{j+1;0}(t) + \frac{t_{j+3} - t}{h_{j+2;1}} \cdot b_{j+2;0}(t) \right)$$

$$b_{j+1;2}(t) = \frac{t - t_{j+1}}{h_{j+1;2}} \cdot \left(\frac{t - t_{j+1}}{h_{j+1;1}} \cdot b_{j+1;0}(t) + \frac{t_{j+3} - t}{h_{j+2;1}} \cdot b_{j+2;0}(t) \right) + \frac{t_{j+4} - t}{h_{j+2;2}} \cdot \left(\frac{t - t_{j+2}}{h_{j+2;1}} \cdot b_{j+2;0}(t) + \frac{t_{j+4} - t}{h_{j+3;1}} \cdot b_{j+3;0}(t) \right)$$

$$b_{j;3}(t) = \frac{t - t_j}{h_{j;3}} \cdot b_{j;2}(t) + \frac{t_{j+4} - t}{h_{j+1;3}} \cdot b_{j+1;2}(t)$$

$$b_{j;3}(t) = \frac{t - t_j}{h_{j;3}} \cdot \left(\frac{t - t_j}{h_{j;2}} \cdot \left(\frac{t - t_j}{h_{j;1}} \cdot b_{j;0}(t) + \frac{t_{j+2} - t}{h_{j+1;1}} \cdot b_{j+1;0}(t) \right) + \frac{t_{j+3} - t}{h_{j+1;2}} \cdot \left(\frac{t - t_{j+1}}{h_{j+1;1}} \cdot b_{j+1;0}(t) + \frac{t_{j+3} - t}{h_{j+2;1}} \cdot b_{j+2;0}(t) \right) \right)$$
$$+ \frac{t_{j+4} - t}{h_{j+1;3}} \cdot \left(\frac{t - t_{j+1}}{h_{j+1;2}} \cdot \left(\frac{t - t_{j+1}}{h_{j+1;1}} \cdot b_{j+1;0}(t) + \frac{t_{j+3} - t}{h_{j+2;1}} \cdot b_{j+2;0}(t) \right) + \frac{t_{j+4} - t}{h_{j+2;2}} \cdot \left(\frac{t - t_{j+2}}{h_{j+2;1}} \cdot b_{j+2;0}(t) + \frac{t_{j+4} - t}{h_{j+3;1}} \cdot b_{j+3;0}(t) \right) \right)$$

Let us evaluate in $b_{j;3}(t_j)$to already have coefficients.

Let's remember that :$s_i(t_i)=\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\circ\begin{pmatrix}a_{i,1;1} & a_{i,1;2} & a_{i,1;3} & a_{i,1;4}\\ a_{i,2;1} & a_{i,2;2} & a_{i,2;3} & a_{i,2;4}\\ a_{i,3;1} & a_{i,3;2} & a_{i,3;3} & a_{i,3;4}\\ a_{i,4;1} & a_{i,4;2} & a_{i,4;3} & a_{i,4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$

$s_i(t_i)=a_{i,1;1}\cdot U_{i-1}+a_{i,1;2}\cdot U_i+a_{i,1;3}\cdot U_{i+1}+a_{i,1;4}\cdot U_{i+2}$

and

$s_i(t_i)=\sum_{j=i-1}^{j+2} b_{j-2;3}(t_i)\cdot U_j$

$s_i(t_i)=b_{i-3;3}(t_i)\cdot U_{i-1}+b_{i-2;3}(t_i)\cdot U_i+b_{i-1;3}(t_i)\cdot U_{i+1}+b_{i;3}(t_i)\cdot U_{i+2}$

So there is a direct link between the coefficients of the matrix and the$b_{j;3}(t_i)$
They are explained below.

Recall that $b_{j;0}(t)$is zero if $t\notin[t_j,t_{j+1}]$.$b_{j;0}(t)=\begin{cases}1 & si\, t_j\le t<t_{j+1}\\ 0 & sinon\end{cases}$

$b_{j;0}(t_i)=1$only if $i=j$. In the case where $t_j=j$, we have$h_{j;k}=k$

$$b_{j;3}(t_i)=\frac{t_i-t_j}{h_{j;3}}\cdot\left(\frac{t_i-t_j}{h_{j;2}}\cdot\left(\frac{t_i-t_j}{h_{j;1}}\cdot b_{j;0}(t_i)+\frac{t_{j+2}-t_i}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)\right)+\frac{t_{j+3}-t_i}{h_{j+1;2}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)+\frac{t_{j+3}-t_i}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)\right)\right)$$
$$+$$
$$\frac{t_{j+4}-t_i}{h_{j+1;3}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;2}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)+\frac{t_{j+3}-t_i}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)\right)+\frac{t_{j+4}-t_i}{h_{j+2;2}}\cdot\left(\frac{t_i-t_{j+2}}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)+\frac{t_{j+4}-t_i}{h_{j+3;1}}\cdot b_{j+3;0}(t_i)\right)\right)$$

<u>First coefficient:</u>

$a_{i,1;1}=b_{i-3;3}(t_i)=$

$$\frac{t_i-t_{i-3}}{h_{i-3;3}}\cdot\left(\frac{t_i-t_{i-3}}{h_{i-3;2}}\cdot\left(\frac{t_i-t_{i-3}}{h_{i-3;1}}\cdot b_{i-3;0}(t_i)+\frac{t_{i-1}-t_i}{h_{i-2;1}}\cdot b_{i-2;0}(t_i)\right)+\frac{t_i-t_i}{h_{i-2;2}}\cdot\left(\frac{t_i-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t_i)+\frac{t_i-t_i}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)\right)\right)$$
$$+$$
$$\frac{t_{i+1}-t_i}{h_{i-2;3}}\cdot\left(\frac{t_i-t_{i-2}}{h_{i-2;2}}\cdot\left(\frac{t_i-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t_i)+\frac{t_i-t_i}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)\right)+\frac{t_{i+1}-t_i}{h_{i-1;2}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)+\frac{t_{i+1}-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)\right)\right)$$

$== a_{i,1;1}$_

$$\frac{t_{i+1}-t_i}{h_{i-2;3}}\cdot\frac{t_{i+1}-t_i}{h_{i-1;2}}\cdot\frac{t_{i+1}-t_i}{h_{i;1}}=\frac{h_{i;1}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}\cdot\frac{h_{i;1}}{h_{i;1}}$$

So$a_{i,1;1}=b_{i-3;3}(t_i)=\frac{h_{i;1}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}$

Verification in the equidistant case where $t_j=j$, $h_{j;k}=k$:

$a_{i,1;1}=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{6}$, that's what we've been waiting for!

<u>Second coefficient:</u>

Recall that $b_{j;3}(t_i)=$

$$\frac{t_i-t_j}{h_{j;3}}\cdot\left(\frac{t_i-t_j}{h_{j;2}}\cdot\left(\frac{t_i-t_j}{h_{j;1}}\cdot b_{j;0}(t_i)+\frac{t_{j+2}-t_i}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)\right)+\frac{t_{j+3}-t_i}{h_{j+1;2}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)+\frac{t_{j+3}-t_i}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)\right)\right)+$$

$$\frac{t_{j+4}-t_i}{h_{j+1;3}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;2}}\cdot\left(\frac{t_i-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t_i)+\frac{t_{j+3}-t_i}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)\right)+\frac{t_{j+4}-t_i}{h_{j+2;2}}\cdot\left(\frac{t_i-t_{j+2}}{h_{j+2;1}}\cdot b_{j+2;0}(t_i)+\frac{t_{j+4}-t_i}{h_{j+3;1}}\cdot b_{j+3;0}(t_i)\right)\right)$$

$a_{i,1;2}=b_{i-2;3}(t_i)=$

$$\frac{t_i-t_{i-2}}{h_{i-2;3}}\cdot\left(\frac{t_i-t_{i-2}}{h_{i-2;2}}\cdot\left(\frac{t_i-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t_i)+\frac{t_i-t_i}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)\right)+\frac{t_{i+1}-t_i}{h_{i-1;2}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)+\frac{t_{i+1}-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)\right)\right)+$$

$$\frac{t_{i+2}-t_i}{h_{i-1;3}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;2}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)+\frac{t_{i+1}-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)\right)+\frac{t_{i+2}-t_i}{h_{i;2}}\cdot\left(\frac{t_i-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)+\frac{t_{i+2}-t_i}{h_{i+1;1}}\cdot b_{i+1;0}(t_i)\right)\right)$$

== $a_{i,1;2}$_

$$\frac{t_i-t_{i-2}}{h_{i-2;3}}\cdot\frac{t_{i+1}-t_i}{h_{i-1;2}}\cdot\frac{t_{i+1}-t_i}{h_{i;1}}+\frac{t_{i+2}-t_i}{h_{i-1;3}}\cdot\frac{t_i-t_{i-1}}{h_{i-1;2}}\cdot\frac{t_{i+1}-t_i}{h_{i;1}}=$$

$$\frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}\cdot\frac{h_{i;1}}{h_{i;1}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}\cdot\frac{h_{i;1}}{h_{i;1}}=$$

So $a_{i,1;2}=b_{i-2;3}(t_i)=\frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}$

Verification in the equidistant case where $t_j=j$, $h_{j;k}=k$:

$a_{i,1;2}=\frac{2}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}=\frac{4}{6}$, that's what we've been waiting for!

<u>Third coefficient:</u>

$a_{i,1;3}=b_{i-1;3}(t_i)=$

$$\frac{t_i-t_{i-1}}{h_{i-1;3}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;2}}\cdot\left(\frac{t_i-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t_i)+\frac{t_{i+1}-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)\right)+\frac{t_{i+2}-t_i}{h_{i;2}}\cdot\left(\frac{t_i-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)+\frac{t_{i+2}-t_i}{h_{i+1;1}}\cdot b_{i+1;0}(t_i)\right)\right)+$$

$$\frac{t_{i+3}-t_i}{h_{i;3}}\cdot\left(\frac{t_i-t_i}{h_{i;2}}\cdot\left(\frac{t_i-t_i}{h_{i;1}}\cdot b_{i;0}(t_i)+\frac{t_{i+2}-t_i}{h_{i+1;1}}\cdot b_{i+1;0}(t_i)\right)+\frac{t_{i+3}-t_i}{h_{i+1;2}}\cdot\left(\frac{t_i-t_{i+1}}{h_{i+1;1}}\cdot b_{i+1;0}(t_i)+\frac{t_{i+3}-t_i}{h_{i+2;1}}\cdot b_{i+2;0}(t_i)\right)\right)$$

== $a_{i,1;3}$_

$$\frac{t_i-t_{i-1}}{h_{i-1;3}}\cdot\frac{t_i-t_{i-1}}{h_{i-1;2}}+0=$$

So $a_{i,1;3}=\frac{h_{i-1;1}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}$

Verification in the equidistant case where $t_j=j$, $h_{j;k}=k$:

$a_{i,1;3}=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$, that's what we've been waiting for!

Fourth coefficient:

$a_{i,1;4}=b_{i;3}(t_i)=$

$0+$

$$\frac{t_{i+4}-t_i}{h_{i+1;3}}\cdot\left(\frac{t_i-t_{i+1}}{h_{i+1;2}}\cdot\left(\frac{t_i-t_{i+1}}{h_{i+1;1}}\cdot b_{i+1;0}(t_i)+\frac{t_{i+3}-t_i}{h_{i+2;1}}\cdot b_{i+2;0}(t_i)\right)+\frac{t_{i+4}-t_i}{h_{i+2;2}}\cdot\left(\frac{t_i-t_{i+2}}{h_{i+2;1}}\cdot b_{i+2;0}(t_i)+\frac{t_{i+4}-t_i}{h_{i+3;1}}\cdot b_{i+3;0}(t_i)\right)\right)$$

$=0$

So $a_{i,1;4}=0$ that's what we expected

A more important check is: $a_{i,1;1}+a_{i,1;2}+a_{i,1;3}+a_{i,1;4}=1$

If this were not the case, a translation of the control points would not result in a translation of the curve. It would therefore depend on the choice of origin.

Useful rule for simplifications:

$h_{i+k;l}+h_{i+m;k-m}=t_{i+k+l}-t_{i+k}+t_{i+k}-t_{i+m}=h_{i+m;k+l-m2}$

$$a_{i,1;1}+a_{i,2;1}+a_{i,3;1}+a_{i,4;1}=\frac{h_{i;1}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}+\frac{h_{i-1;1}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}+0$$

$$\frac{h_{i;1}\cdot h_{i;1}\cdot h_{i-1;3}+h_{i-2;2}\cdot h_{i;1}\cdot h_{i-1;3}+h_{i;2}\cdot h_{i-1;1}\cdot h_{i-2;3}+h_{i-1;1}\cdot h_{i-1;1}\cdot h_{i-2;3}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i-1;3}}$$

$$\frac{h_{i;1}\cdot h_{i-1;3}\cdot(h_{i;1}+h_{i-2;2})+h_{i-1;1}\cdot h_{i-2;3}\cdot(h_{i;2}+h_{i-1;1})}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i-1;3}}$$

$$\frac{h_{i;1}\cdot h_{i-1;3}\cdot(h_{i-2;3})+h_{i-1;1}\cdot h_{i-2;3}\cdot(h_{i-1;3})}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i-1;3}}$$

$$\frac{h_{i;1}+h_{i-1;1}}{h_{i-1;2}}\ \frac{h_{i-1;2}}{h_{i-1;2}}=1$$

It's magic ! That works !

Let's find the coefficients of the first column, $a_{i,4;1};a_{i,3;1};a_{i,2;1};a_{i,1;1}$

These are the coefficients of the polynomial of $b_{i-3;3}(t)$, written as a polynomial in $(t-t_i)^3$.

Let's remember that :

$$b_{j;3}(t)=\frac{t-t_j}{h_{j;3}}\cdot\left(\frac{t-t_j}{h_{j;2}}\cdot\left(\frac{t-t_j}{h_{j;1}}\cdot b_{j;0}(t)+\frac{t_{j+2}-t}{h_{j+1;1}}\cdot b_{j+1;0}(t)\right)+\frac{t_{j+3}-t}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)\right)$$

$$+\frac{t_{j+4}-t}{h_{j+1;3}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)+\frac{t_{j+4}-t}{h_{j+2;2}}\cdot\left(\frac{t-t_{j+2}}{h_{j+2;1}}\cdot b_{j+2;0}(t)+\frac{t_{j+4}-t}{h_{j+3;1}}\cdot b_{j+3;0}(t)\right)\right)$$

and

$s_i(t)=b_{i-3;3}(t)\cdot U_{i-1}+b_{i-2;3}(t)\cdot U_i+b_{i-1;3}(t)\cdot U_{i+1}+b_{i;3}(t)\cdot U_{i+2}$

So

$a_{i,4;1}$ is the factor $(t-t_i)^3$ of $b_{i-3;3}(t)$

$a_{i,3;1}$ is the factor $(t-t_i)^2$ of $b_{i-3;3}(t)$

$a_{i,2;1}$ is the factor $(t-t_i)^1$ of $b_{i-3;3}(t)$

$a_{i,1;1}$ is the factor $(t-t_i)^0$ of $b_{i-3;3}(t)$

$$b_{i-3;3}(t)=\frac{t-t_{i-3}}{h_{i-3;3}}\cdot\left(\frac{t-t_{i-3}}{h_{i-3;2}}\cdot\left(\frac{t-t_{i-3}}{h_{i-3;1}}\cdot b_{i-3;0}(t)+\frac{t_{i-1}-t}{h_{i-2;1}}\cdot b_{i-2;0}(t)\right)+\frac{t_i-t}{h_{i-2;2}}\cdot\left(\frac{t-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t)+\frac{t_i-t}{h_{i-1;1}}\cdot b_{i-1;0}(t)\right)\right)$$
$$+\frac{t_{i+1}-t}{h_{i-2;3}}\cdot\left(\frac{t-t_{i-2}}{h_{i-2;2}}\cdot\left(\frac{t-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t)+\frac{t_i-t}{h_{i-1;1}}\cdot b_{i-1;0}(t)\right)+\frac{t_{i+1}-t}{h_{i-1;2}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t)+\frac{t_{i+1}-t}{h_{i;1}}\cdot b_{i;0}(t)\right)\right)$$

$t\in[t_i,t_{i+1}]$, so $b_{j;0}(t)=1$only if $j=i$.
This simplifies:

$$b_{i-3;3}(t)=0+\frac{t_{i+1}-t}{h_{i-2;3}}\cdot\left(0+\frac{t_{i+1}-t}{h_{i-1;2}}\cdot\left(0+\frac{t_{i+1}-t}{h_{i;1}}\right)\right)$$

$$b_{i-3;3}(t)=-\frac{t-t_{i+1}}{h_{i-2;3}}\cdot\frac{t-t_{i+1}}{h_{i-1;2}}\cdot\frac{t-t_{i+1}}{h_{i;1}}$$
$$b_{i-3;3}(t)=-\frac{(t-t_i)+t_i-t_{i+1}}{h_{i-2;3}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i-1;2}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i;1}}$$
$$b_{i-3;3}(t)=-\frac{(t-t_i)-h_{i;1}}{h_{i-2;3}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i-1;2}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i;1}}$$
$$b_{i-3;3}(t)=-\frac{((t-t_i)-h_{i;1})^3}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}$$
$$b_{i-3;3}(t)=$$
$$-\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}\cdot(t-t_i)^3+\frac{3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}\cdot(t-t_i)^2-\frac{3h_{i;1}^2}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}\cdot(t-t_i)+\frac{h_{i;1}^3}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}$$

$$b_{i-3;3}(t)=-\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}\cdot(t-t_i)^3+\frac{3}{h_{i-2;3}\cdot h_{i-1;2}}\cdot(t-t_i)^2-\frac{3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}}\cdot(t-t_i)+\frac{h_{i;1}^2}{h_{i-2;3}\cdot h_{i-1;2}}$$

The 4 factors give the 4 coefficients: $a_{i,4;1}$; $a_{i,3;1}$; $a_{i,2;1}$and$a_{i,1;1}$
We find the value of$a_{i,1;1}$
So we have :

$a_{i,1;1}=\frac{h_{i;1}^2}{h_{i-2;3}\cdot h_{i-1;2}}$ $=\frac{1}{6}$in the equidistant case where $t_j=j, h_{j;k}=k$

$a_{i,2;1}=-\frac{3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}}$ = -3 in the equidistant case

$a_{i,3;1}=\frac{3}{h_{i-2;3}\cdot h_{i-1;2}}$ $=\frac{3}{6}$in the equidistant case

$a_{i,4;1}=-\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}$ $=-\frac{1}{6}$in the equidistant case

Let's find the coefficients of the second column, $a_{i,4;2}; a_{i,3;2}; a_{i,2;2}; a_{i,1;2}$
These are the coefficients of the polynomial of $b_{i-2;3}(t)$, written as a polynomial in $(t-t_i)^3$.

Let's remember that :

$$b_{j;3}(t)=\frac{t-t_j}{h_{j;3}}\cdot\left(\frac{t-t_j}{h_{j;2}}\cdot\left(\frac{t-t_j}{h_{j;1}}\cdot b_{j;0}(t)+\frac{t_{j+2}-t}{h_{j+1;1}}\cdot b_{j+1;0}(t)\right)+\frac{t_{j+3}-t}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)\right)$$
$$+\frac{t_{j+4}-t}{h_{j+1;3}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)+\frac{t_{j+4}-t}{h_{j+2;2}}\cdot\left(\frac{t-t_{j+2}}{h_{j+2;1}}\cdot b_{j+2;0}(t)+\frac{t_{j+4}-t}{h_{j+3;1}}\cdot b_{j+3;0}(t)\right)\right)$$

and

$$s_i(t)=b_{i-3;3}(t)\cdot U_{i-1}+b_{i-2;3}(t)\cdot U_i+b_{i-1;3}(t)\cdot U_{i+1}+b_{i;3}(t)\cdot U_{i+2}$$

So
$a_{i,4;2}$ is the factor $(t-t_i)^3$ of $b_{i-2;3}(t)$
$a_{i,3;2}$ is the factor $(t-t_i)^2$ of $b_{i-2;3}(t)$
$a_{i,2;2}$ is the factor $(t-t_i)^1$ of $b_{i-2;3}(t)$
$a_{i,1;2}$ is the factor $(t-t_i)^0$ of $b_{i-2;3}(t)$

$$b_{i-2;3}(t)=\frac{t-t_{i-2}}{h_{i-2;3}}\cdot\left(\frac{t-t_{i-2}}{h_{i-2;2}}\cdot\left(\frac{t-t_{i-2}}{h_{i-2;1}}\cdot b_{i-2;0}(t)+\frac{t_i-t}{h_{i-1;1}}\cdot b_{i-1;0}(t)\right)+\frac{t_{i+1}-t}{h_{i-1;2}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t)+\frac{t_{i+1}-t}{h_{i;1}}\cdot b_{i;0}(t)\right)\right)$$
$$+\frac{t_{i+2}-t}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t)+\frac{t_{i+1}-t}{h_{i;1}}\cdot b_{i;0}(t)\right)+\frac{t_{i+2}-t}{h_{i;2}}\cdot\left(\frac{t-t_i}{h_{i;1}}\cdot b_{i;0}(t)+\frac{t_{i+2}-t}{h_{i+1;1}}\cdot b_{i+1;0}(t)\right)\right)$$

$t\in[t_i,t_{i+1}]$, so $b_{j;0}(t)=1$ only if $j=i$.
This simplifies:

$$b_{i-2;3}(t)=\frac{t-t_{i-2}}{h_{i-2;3}}\cdot\frac{t_{i+1}-t}{h_{i-1;2}}\cdot\frac{t_{i+1}-t}{h_{i;1}}+\frac{t_{i+2}-t}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\frac{t_{i+1}-t}{h_{i;1}}+\frac{t_{i+2}-t}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}\right)$$
$$b_{i-2;3}(t)=\frac{t-t_{i-2}}{h_{i-2;3}}\cdot\frac{t-t_{i+1}}{h_{i-1;2}}\cdot\frac{t-t_{i+1}}{h_{i;1}}+\frac{t-t_{i+2}}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\frac{t-t_{i+1}}{h_{i;1}}+\frac{t-t_{i+2}}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}\right)$$

$$b_{i-2;3}(t)=\frac{(t-t_i)+t_i-t_{i-2}}{h_{i-2;3}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i-1;2}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i;1}}+$$
$$\frac{(t-t_i)+t_i-t_{i+2}}{h_{i-1;3}}\cdot\left(\frac{(t-t_i)+t_i-t_{i-1}}{h_{i-1;2}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i;1}}+\frac{(t-t_i)+t_i-t_{i+2}}{h_{i;2}}\cdot\frac{(t-t_i)+t_i-t_i}{h_{i;1}}\right)$$

$$b_{i-2;3}(t)=\frac{(t-t_i)+h_{i-2;2}}{h_{i-2;3}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i-1;2}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i;1}}+$$
$$\frac{(t-t_i)-h_{i;2}}{h_{i-1;3}}\cdot\left(\frac{(t-t_i)+h_{i-1;1}}{h_{i-1;2}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i;1}}+\frac{(t-t_i)-h_{i;2}}{h_{i;2}}\cdot\frac{(t-t_i)}{h_{i;1}}\right)$$

$$b_{i-2;3}(t)=(t-t_i)^3\cdot\left(\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+(t-t_i)^2\cdot\left(\frac{h_{i-2;2}-h_{i;1}-h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}-h_{i;2}-h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}-\frac{h_{i;2}+h_{i;2}}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+(t-t_i)\cdot\left(\frac{h_{i;1}^2-2h_{i;1}\cdot h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}^2}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+\frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}$$

The 4 factors give the 4 coefficients: $a_{i,4;2}$; $a_{i,3;2}$; $a_{i,2;2}$and$a_{i,1;2}$
We find the value of$a_{i,1;2}$
So we have :

$a_{i,1;2}=\frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}}\quad=\frac{4}{6}$in the equidistant case where $t_j=j,h_{j;k}=k$

$a_{i,2;2}=\frac{h_{i;1}-2h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}}{h_{i-1;3}\cdot h_{i;1}}=0$ if equidistant

$a_{i,3;2}=\frac{h_{i-2;2}-h_{i;1}-h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}-h_{i;2}-h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}-\frac{2}{h_{i-1;3}\cdot h_{i;1}}=-\frac{6}{6}$in the equidistant case

$a_{i,4;2}=\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}\qquad=\frac{3}{6}$in the equidistant case

<u>Let's find the coefficients of the third column,</u> $a_{i,4;3}; a_{i,3;3}; a_{i,2;3}; a_{i,1;3}$

These are the coefficients of the polynomial of $b_{i-1;3}(t)$, written as a polynomial in $(t-t_i)^3$.

Let's remember that :

$$b_{j;3}(t)=\frac{t-t_j}{h_{j;3}}\cdot\left(\frac{t-t_j}{h_{j;2}}\cdot\left(\frac{t-t_j}{h_{j;1}}\cdot b_{j;0}(t)+\frac{t_{j+2}-t}{h_{j+1;1}}\cdot b_{j+1;0}(t)\right)+\frac{t_{j+3}-t}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)\right)$$
$$+\frac{t_{j+4}-t}{h_{j+1;3}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;2}}\cdot\left(\frac{t-t_{j+1}}{h_{j+1;1}}\cdot b_{j+1;0}(t)+\frac{t_{j+3}-t}{h_{j+2;1}}\cdot b_{j+2;0}(t)\right)+\frac{t_{j+4}-t}{h_{j+2;2}}\cdot\left(\frac{t-t_{j+2}}{h_{j+2;1}}\cdot b_{j+2;0}(t)+\frac{t_{j+4}-t}{h_{j+3;1}}\cdot b_{j+3;0}(t)\right)\right)$$

and

$$s_i(t)=b_{i-3;3}(t)\cdot U_{i-1}+b_{i-2;3}(t)\cdot U_i+b_{i-1;3}(t)\cdot U_{i+1}+b_{i;3}(t)\cdot U_{i+2}$$

So

$a_{i,4;3}$is the factor $(t-t_i)^3$of$b_{i-1;3}(t)$

$a_{i,3;3}$is the factor $(t-t_i)^2$of$b_{i-1;3}(t)$

$a_{i,2;3}$is the factor $(t-t_i)^1$of$b_{i-1;3}(t)$

$a_{i,1;3}$is the factor $(t-t_i)^0$of$b_{i-1;3}(t)$

$$b_{i-1;3}(t)=\frac{t-t_{i-1}}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;1}}\cdot b_{i-1;0}(t)+\frac{t_{i+1}-t}{h_{i;1}}\cdot b_{i;0}(t)\right)+\frac{t_{i+2}-t}{h_{i;2}}\cdot\left(\frac{t-t_i}{h_{i;1}}\cdot b_{i;0}(t)+\frac{t_{i+2}-t}{h_{i+1;1}}\cdot b_{i+1;0}(t)\right)\right)$$
$$+\frac{t_{i+3}-t}{h_{i;3}}\cdot\left(\frac{t-t_i}{h_{i;2}}\cdot\left(\frac{t-t_i}{h_{i;1}}\cdot b_{i;0}(t)+\frac{t_{i+2}-t}{h_{i+1;1}}\cdot b_{i+1;0}(t)\right)+\frac{t_{i+3}-t}{h_{i+1;2}}\cdot\left(\frac{t-t_{i+1}}{h_{i+1;1}}\cdot b_{i+1;0}(t)+\frac{t_{i+3}-t}{h_{i+2;1}}\cdot b_{i+2;0}(t)\right)\right)$$

$t\in[t_i,t_{i+1}]$, so $b_{j;0}(t)=1$only if $j=i$.

This simplifies:

$$b_{i-1;3}(t)=\frac{t-t_{i-1}}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\frac{t_{i+1}-t}{h_{i;1}}+\frac{t_{i+2}-t}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}\right)+\frac{t_{i+3}-t}{h_{i;3}}\cdot\frac{t-t_i}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}$$

$$b_{i-1;3}(t)=-\frac{t-t_{i-1}}{h_{i-1;3}}\cdot\left(\frac{t-t_{i-1}}{h_{i-1;2}}\cdot\frac{t-t_{i+1}}{h_{i;1}}+\frac{t-t_{i+2}}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}\right)-\frac{t-t_{i+3}}{h_{i;3}}\cdot\frac{t-t_i}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}$$

$$b_{i-1;3}(t)=-\frac{(t-t_i)+t_i-t_{i-1}}{h_{i-1;3}}\cdot\left(\frac{(t-t_i)+t_i-t_{i-1}}{h_{i-1;2}}\cdot\frac{(t-t_i)+t_i-t_{i+1}}{h_{i;1}}+\frac{(t-t_i)+t_i-t_{i+2}}{h_{i;2}}\cdot\frac{(t-t_i)}{h_{i;1}}\right)$$
$$-\frac{(t-t_i)+t_i-t_{i+3}}{h_{i;3}}\cdot\frac{(t-t_i)}{h_{i;2}}\cdot\frac{(t-t_i)}{h_{i;1}}$$

$$b_{i-1;3}(t)=-\frac{(t-t_i)+h_{i-1;1}}{h_{i-1;3}}\cdot\left(\frac{(t-t_i)+h_{i-1;1}}{h_{i-1;2}}\cdot\frac{(t-t_i)-h_{i;1}}{h_{i;1}}+\frac{(t-t_i)-h_{i;2}}{h_{i;2}}\cdot\frac{(t-t_i)}{h_{i;1}}\right)$$
$$-\frac{(t-t_i)-h_{i;3}}{h_{i;3}}\cdot\frac{(t-t_i)}{h_{i;2}}\cdot\frac{(t-t_i)}{h_{i;1}}$$

$$b_{i-1;3}(t)=(t-t_i)^3\cdot\left(\frac{-1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{-1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{-1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+(t-t_i)^2\cdot\left(\frac{h_{i;1}-2h_{i-1;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}-h_{i-1;1}}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{h_{i;3}}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+(t-t_i)\cdot\left(\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}\cdot h_{i;2}}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$
$$+\left(\frac{h_{i-1;1}^2\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}\right)$$

The 4 factors give the 4 coefficients: $a_{i,4;3}$; $a_{i,3;3}$; $a_{i,2;3}$and$a_{i,1;3}$
We find the value of$a_{i,1;3}$
So we have :

$a_{i,1;3}=\frac{h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}}$ $=\frac{1}{6}$in the equidistant case where $t_j=j, h_{j;k}=k$

$a_{i,2;3}=\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}}{h_{i-1;3}\cdot h_{i;1}}=\frac{3}{6}$if equidistant

$a_{i,3;3}=\frac{h_{i;1}-2h_{i-1;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}-h_{i-1;1}}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{1}{h_{i;2}\cdot h_{i;1}}=\frac{3}{6}$in the equidistant case

$a_{i,4;3}=\frac{-1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{-1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{-1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}$ $=-\frac{3}{6}$in the equidistant case

<u>Let's find the coefficients of the fourth column,</u> $a_{i,4;4}; a_{i,3;4}; a_{i,2;4}; a_{i,1;4}$
These are the coefficients of the polynomial of $b_{i;3}(t)$, written as a polynomial in $(t-t_i)^3$.

Let's remember that :

$$b_{i;4}(t)=\frac{t-t_i}{h_{i;3}}\cdot\left(\frac{t-t_i}{h_{i;2}}\cdot\left(\frac{t-t_i}{h_{i;1}}\cdot b_{i;0}(t)+\frac{t_{i+2}-t}{h_{i+1;1}}\cdot b_{i+1;0}(t)\right)+\frac{t_{i+3}-t}{h_{i+1;2}}\cdot\left(\frac{t-t_{i+1}}{h_{i+1;1}}\cdot b_{i+1;0}(t)+\frac{t_{i+3}-t}{h_{i+2;1}}\cdot b_{i+2;0}(t)\right)\right)$$

$$+\frac{t_{i+4}-t}{h_{i+1;3}}\cdot\left(\frac{t-t_{i+1}}{h_{i+1;2}}\cdot\left(\frac{t-t_{i+1}}{h_{i+1;1}}\cdot b_{i+1;0}(t)+\frac{t_{i+3}-t}{h_{i+2;1}}\cdot b_{i+2;0}(t)\right)+\frac{t_{i+4}-t}{h_{i+2;2}}\cdot\left(\frac{t-t_{i+2}}{h_{i+2;1}}\cdot b_{i+2;0}(t)+\frac{t_{i+4}-t}{h_{i+3;1}}\cdot b_{i+3;0}(t)\right)\right)$$

and

$$s_i(t)=b_{i-3;3}(t)\cdot U_{i-1}+b_{i-2;3}(t)\cdot U_i+b_{i-1;3}(t)\cdot U_{i+1}+b_{i;3}(t)\cdot U_{i+2}$$

$t\in[t_i,t_{i+1}]$, so $b_{j;0}(t)=1$only if $j=i$.
This simplifies:

$$b_{i;4}(t)=\frac{t-t_i}{h_{i;3}}\cdot\frac{t-t_i}{h_{i;2}}\cdot\frac{t-t_i}{h_{i;1}}\quad b_{i;4}(t)=(t-t_i)^3\cdot\left(\frac{1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}\right)$$

$a_{i,4;4}$is the factor of $(t-t_i)^3$of $b_{i;3}(t)$, so

$$a_{i,4;4}=\frac{1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}\qquad=\frac{1}{6}$$ in the equidistant case where $t_j=j, h_{j;k}=k$

$a_{i,3;4}=0$; $a_{i,2;4}=0$;$a_{i,1;4}=0$

Summary - result - conclusion of previous calculations.

$$s_i(t)=\begin{pmatrix}1 & t-t_i & (t-t_i)^2 & (t-t_i)^3\end{pmatrix}\circ A_i\circ\begin{pmatrix}U_{i-1}\\U_i\\U_{i+1}\\U_{i+2}\end{pmatrix}\text{with}A_i=\begin{pmatrix}a_{i,1;1} & a_{i,1;2} & a_{i,1;3} & a_{i,1;4}\\a_{i,2;1} & a_{i,2;2} & a_{i,2;3} & a_{i,2;4}\\a_{i,3;1} & a_{i,3;2} & a_{i,3;3} & a_{i,3;4}\\a_{i,4;1} & a_{i,4;2} & a_{i,4;3} & a_{i,4;4}\end{pmatrix}$$

The curve (B-spline) defined by $s(t)=s_i(t)$ for $t\in[t_i..t_{i+1}]$,
$h_{j;k}=t_{j+k}-t_j$, so $h_j=h_{j;1}=t_{j+1}-t_j$

$$A_i=\begin{pmatrix}\frac{h_{i;1}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}} & \frac{h_{i-2;2}}{h_{i-2;3}}\cdot\frac{h_{i;1}}{h_{i-1;2}}+\frac{h_{i;2}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}} & \frac{h_{i-1;1}}{h_{i-1;3}}\cdot\frac{h_{i-1;1}}{h_{i-1;2}} & 0\\ \frac{-3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}} & a_{i,2;2} & a_{i,2;3} & 0\\ \frac{3}{h_{i-2;3}\cdot h_{i-1;2}} & a_{i,3;2} & a_{i,3;3} & 0\\ \frac{-1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}} & a_{i,4;2} & a_{i,4;3} & \frac{1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}\end{pmatrix}$$

$$a_{i,2;2}=\frac{h_{i;1}-2h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}}{h_{i-1;3}\cdot h_{i;1}}$$
$$a_{i,3;2}=\frac{h_{i-2;2}-2h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}-h_{i;2}-h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}-\frac{2}{h_{i-1;3}\cdot h_{i;1}}$$
$$a_{i,4;2}=\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}$$
$$a_{i,2;3}=\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}}{h_{i-1;3}\cdot h_{i;1}}$$
$$a_{i,3;3}=\frac{h_{i;1}-2h_{i-1;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}-h_{i-1;1}}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{1}{h_{i;2}\cdot h_{i;1}}$$
$$a_{i,4;3}=\frac{-1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{-1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{-1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}$$

If we want to have $s_i(t_i+\tau\cdot h_i)=\begin{pmatrix}1 & \tau & \tau^2 & \tau^3\end{pmatrix}\circ\begin{pmatrix}\alpha_{i,1;1} & \alpha_{i,1;2} & \alpha_{i,1;3} & \alpha_{i,1;4}\\\alpha_{i,2;1} & \alpha_{i,2;2} & \alpha_{i,2;3} & \alpha_{i,2;4}\\\alpha_{i,3;1} & \alpha_{i,3;2} & \alpha_{i,3;3} & \alpha_{i,3;4}\\\alpha_{i,4;1} & \alpha_{i,4;2} & \alpha_{i,4;3} & \alpha_{i,4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\U_i\\U_{i+1}\\U_{i+2}\end{pmatrix}$

We have the relation: $\alpha_{i,j;k}=h_i^{(j-1)}\cdot a_{i,j;k}$ $\quad\tau=\frac{t-t_i}{h_i}$ $\quad i = 1..n$; j = 1..4; k = 1 .. 4

For i = 1 .. n , we have:
$V_i=s(t_i)=s_i(t_i)=a_{i,1;1}\cdot U_{i-1}+a_{i,1;2}\cdot U_i+a_{i,1;3}\cdot U_{i+1}$
$p_i=s'(t_i)=s'_i(t_i)=a_{i,2;1}\cdot U_{i-1}+a_{i,2;2}\cdot U_i+a_{i,2;3}\cdot U_{i+1}$
$M_i=s''(t_i)=s''_i(t_i)=a_{i,3;1}\cdot U_{i-1}+a_{i,3;2}\cdot U_i+a_{i,3;3}\cdot U_{i+1}$
The relation between the V_i and the U_i can be written as a matrix, which makes it possible to pass control points from the B-spline to the Math-spline and vice versa.

How to determine the two control points V_0 and V_{n+1} of the Math-spline to make the two curves coincide?
Slope at start = $p_1 = s'(t_1) = s'_1(t_1) = a_{1,2;1} \cdot U_0 + a_{1,2;2} \cdot U_1 + a_{1,2;3} \cdot U_2$.
Slope on arrival = $p_n = s'(t_n) = s'_n(t_n) = a_{n,2;1} \cdot U_{n-1} + a_{n,2;2} \cdot U_n + a_{n,2;3} \cdot U_{n+1}$.
$s'_n(t)$ is only defined at $t = t_n$ and $s'_n(t_n)$ is easier to compute than $s'_{n-1}(t_n)$.
For the Math-spline, these two slopes are free to choose.
It remains to define a link between these two slopes and the two values V_0 and V_{n+1}.
We want the following conditions to be fulfilled:
1) $p_1 = a_{1,2;1} \cdot U_0 + a_{1,2;2} \cdot U_1 + a_{1,2;3} \cdot U_2$
2) $V_0 = \alpha \cdot U_0 + \beta \cdot U_1 + \gamma \cdot U_2$, with $\alpha ; \beta ; \gamma$ free
3) $V_1 = a_{1,1;1} \cdot U_0 + a_{1,1;2} \cdot U_1 + a_{1,1;3} \cdot U_2$
4) $p_1 = \lambda \cdot V_1 - \mu \cdot V_0$, with $\lambda ; \mu$ free
We therefore want to express V_0 as a function of U_0, U_1 and U_2, which makes it possible to determine p_1 which satisfies condition 1), which describes the derivative at the start of the curve.

From these 4 equalities, let p_1, V_0 and V_1 disappear.
1) = 4) and substitute 2) and 3), to obtain:

$$a_{1,2;1} \cdot U_0 + a_{1,2;2} \cdot U_1 + a_{1,2;3} \cdot U_2 = \lambda \cdot a_{1,1;1} \cdot U_0 + \lambda \cdot a_{1,1;2} \cdot U_1 + \lambda \cdot a_{1,1;3} \cdot U_2 - \mu \cdot \alpha \cdot U_0 - \mu \cdot \beta \cdot U_1 - \mu \cdot \gamma \cdot U_2$$

By rearranging the terms and highlighting the U_i:
$U_0 \cdot (a_{1,2;1} - \lambda \cdot a_{1,1;1} + \mu \cdot \alpha) + U_1 \cdot (a_{1,2;2} - \lambda \cdot a_{1,1;2} + \mu \cdot \beta) + U_2 \cdot (a_{1,2;3} - \lambda \cdot a_{1,1;3} + \mu \cdot \gamma) = 0$

We want the equality to be true regardless of the values of U_i, so we must:

$a_{1,2;1} - \lambda \cdot a_{1,1;1} + \mu \cdot \alpha = 0$ and $\alpha = \frac{\lambda}{\mu} \cdot a_{1,1;1} - \frac{1}{\mu} \cdot a_{1,2;1}$

$a_{1,2;2} - \lambda \cdot a_{1,1;2} + \mu \cdot \beta = 0$ and $\beta = \frac{\lambda}{\mu} \cdot a_{1,1;2} - \frac{1}{\mu} \cdot a_{1,2;2}$

$a_{1,2;3} - \lambda \cdot a_{1,1;3} + \mu \cdot \gamma = 0$ $\gamma = \frac{\lambda}{\mu} \cdot a_{1,1;3} - \frac{1}{\mu} \cdot a_{1,2;3}$

A natural choice is: $\lambda = \mu = 1$. With this choice, we get: $p_1 = V_1 - V_0$ and
$\alpha = a_{1,1;1} - a_{1,2;1}$; $\beta = a_{1,1;2} - a_{1,2;2}$; $\gamma = a_{1,1;3} - a_{1,2;3}$.

So : $V_0 = (a_{1,1;1} - a_{1,2;1}) \cdot U_0 + (a_{1,1;2} - a_{1,2;2}) \cdot U_1 + (a_{1,1;3} - a_{1,2;3}) \cdot U_2$

Let's do similar calculations to determine V_{n+1}.

We want the following conditions to be fulfilled:
1) $p_n = s'(t_n) = s'_{n-1}(t_n) = s'_n(t_n) = a_{n,2;1} \cdot U_{n-1} + a_{n,2;2} \cdot U_n + a_{n,2;3} \cdot U_{n+1}$
2) $V_{n+1} = \alpha \cdot U_{n-1} + \beta \cdot U_n + \gamma \cdot U_{n+1}$, with $\alpha ; \beta ; \gamma$ free
3) $V_n = a_{n,1;1} \cdot U_{n-1} + a_{n,1;2} \cdot U_n + a_{n,1;3} \cdot U_{n+1}$
4) $p_n = \mu \cdot V_{n+1} - \lambda \cdot V_n$, with $\lambda ; \mu$ free
We therefore want to express V_{n+1} as a function of U_{n-1}, U_n and U_{n+1}, which makes it possible to determine p_n which satisfies condition 1), which describes the derivative at the end of the curve.

From these 4 equalities, let p_n, V_n and V_{n+1} disappear.
1) = 4) and substitute 2) and 3), to obtain:

$$a_{n,2;1} \cdot U_{n-1} + a_{n,2;2} \cdot U_n + a_{n,2;3} \cdot U_{n+1} = \mu \cdot \alpha \cdot U_{n-1} + \mu \cdot \beta \cdot U_n + \mu \cdot \gamma \cdot U_{n+1} - \lambda \cdot a_{n,1;1} \cdot U_{n-1} - \lambda \cdot a_{n,1;2} \cdot U_n - \lambda \cdot a_{n,1;3} \cdot U$$

By rearranging the terms and highlighting the U_i:

$$U_{n-1} \cdot \left(\frac{1}{2} - \lambda \cdot \frac{1}{6} + \mu \cdot \alpha \right) + U_n \cdot \left(\mu \cdot \beta - \lambda \cdot \frac{4}{6} \right) + U_{n+1} \cdot \left(\mu \cdot \gamma - \lambda \cdot \frac{1}{6} - \frac{1}{2} \right) = 0$$

$$U_{n-1} \cdot \left(a_{n,2;1} + \lambda \cdot a_{n,1;1} - \mu \cdot \alpha \right) + U_n \cdot \left(a_{n,2;2} + \lambda \cdot a_{n,1;2} - \mu \cdot \beta \right) + U_{n+1} \cdot \left(a_{n,2;3} + \lambda \cdot a_{n,1;3} - \mu \cdot \gamma \right) = 0$$

We want the equality to be true regardless of the values of U_i, so we must:

$a_{n,2;1} + \lambda \cdot a_{n,1;1} - \mu \cdot \alpha = 0$ and $\quad \alpha = \frac{\lambda}{\mu} \cdot a_{n,1;1} + \frac{1}{\mu} \cdot a_{n,2;1}$

$a_{n,2;2} + \lambda \cdot a_{n,1;2} - \mu \cdot \beta = 0$ and $\quad \beta = \frac{\lambda}{\mu} \cdot a_{n,1;2} + \frac{1}{\mu} \cdot a_{n,2;2}$

$a_{n,2;3} + \lambda \cdot a_{n,1;3} - \mu \cdot \gamma = 0 \quad \gamma = \frac{\lambda}{\mu} \cdot a_{n,1;3} + \frac{1}{\mu} \cdot a_{n,2;3}$

A natural choice is: $\lambda = \mu = 1$. With this choice, we get: $p_n = V_{n+1} - V_n$ and
$\alpha = a_{n,1;1} + a_{n,2;1}$; $\beta = a_{n,1;2} + a_{n,2;2}$; $\gamma = a_{n,1;3} + a_{n,2;3}$.

So : $V_{n+1} = \left(a_{n,1;1} + a_{n,2;1} \right) \cdot U_{n-1} + \left(a_{n,1;2} + a_{n,2;2} \right) \cdot U_n + \left(a_{n,1;3} + a_{n,2;3} \right) \cdot U_{n+1}$

Writing in matrix form of the transition from U_i to V_i.

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_{n-1} \\ V_n \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} a_{0,1;1} & a_{0,1;2} & a_{0,1;3} & 0 & 0 & 0 & 0 \\ a_{1,1;1} & a_{1,1;2} & a_{1,1;3} & 0 & 0 & 0 & 0 \\ 0 & a_{2,1;1} & a_{2,1;2} & a_{2,1;3} & 0 & 0 & 0 \\ 0 & 0 & a_{3,1;1} & a_{3,1;2} & a_{3,1;3} & 0 & 0 \\ 0 & 0 & 0 & a_{n-1,1;1} & a_{n-1,1;2} & a_{n-1,1;3} & 0 \\ 0 & 0 & 0 & 0 & a_{n,1;1} & a_{n,1;2} & a_{n,1;3} \\ 0 & 0 & 0 & 0 & a_{n+1,1;1} & a_{n+1,1;2} & a_{n+1,1;3} \end{pmatrix} \circ \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_{n-1} \\ U_n \\ U_{n+1} \end{pmatrix}$$

Here, $n = 5$. There are 5 waypoints and 7 checkpoints.

$a_{0,1;1}=a_{1,1;1}-a_{1,2;1}$; $a_{0,1;2}=a_{1,1;2}-a_{1,2;2}$;$a_{0,1;3}=a_{1,1;3}-a_{1,2;3}$

$a_{n+1,1;1}=a_{n,1;1}+a_{n,2;1}$; $a_{n+1,1;2}=a_{n,1;2}+a_{n,2;2}$;$a_{n+1,1;3}=a_{n,1;3}+a_{n,2;3}$

So the transition from the B-spline control points to the Math-spline control points is:

$V_0=(a_{1,1;1}-a_{1,2;1})\cdot U_0+(a_{1,1;2}-a_{1,2;2})\cdot U_1+(a_{1,1;3}-a_{1,2;3})\cdot U_2$;

$V_{n+1}=(a_{n,1;1}+a_{n,2;1})\cdot U_{n-1}+(a_{n,1;2}+a_{n,2;2})\cdot U_n+(a_{n,1;3}+a_{n,2;3})\cdot U_{n+1}$

$V_i=a_{i,1;1}\cdot U_{i-1}+a_{i,1;2}\cdot U_i+a_{i,1;3}\cdot U_{i+1}$, for $i = 1..\ n$

Passing from Math-spline control points to B-spline control points.

Going from Math-spline control points to B-spline control points requires solving the system of equations. Since the matrix is tri-diagonal with dominant diagonal, the calculation is quite fast. The V_i are given, we seek the U_i.

To get a tri-diagonal matrix, let's combine the first two rows and the last two.

Substitutions:

$a_{0,1;1}=a_{0,1;1}-a_{1,1;1}\cdot\frac{a_{0,1;3}}{a_{1,1;3}}$; $a_{0,1;2}=a_{0,1;2}-a_{1,1;2}\cdot\frac{a_{0,1;3}}{a_{1,1;3}}$; $a_{0,1;3}=0$;$V_0=V_0-V_1\cdot\frac{a_{0,1;3}}{a_{1,1;3}}$

$a_{n+1,1;1}=0$; $a_{n+1,1;2}=a_{n+1,1;2}-a_{n,1;2}\cdot\frac{a_{n+1,1;1}}{a_{n,1;1}}$; $a_{n+1,1;3}=a_{n+1,1;3}-a_{n,1;3}\cdot\frac{a_{n+1,1;1}}{a_{n,1;1}}$;$V_{n+1}=V_{n+1}-V_n\cdot\frac{a_{n+1,1;1}}{a_{n,1;1}}$

Resolution :

$lft_0=0$; $diag_0=a_{0,1;1}$; $rigt_0=a_{0,1;2}$; $q_0=V_0$

$lft_{n+1}=a_{n+1,1;2}$; $diag_{n+1}=a_{n+1,1;3}$; $rigt_{n+1}=0$; $q_{n+1}=V_{n+1}$(substituted values)

$lft_i=a_{i,1;1}$; $diag_i=a_{i,1;2}$; $rigt_{n+1}=0$; $q_i=V_i$, for $i = 1..\ n$

for i=1 to n +1 do $diag_i=diag_i-\frac{lft_i}{diag_{i-1}}\cdot rigt_{i-1}$ and $q_i=q_i-\frac{lft_i}{diag_{i-1}}\cdot q_{i-1}$

$U_{n+1}=\frac{q_{n+1}}{diag_{n+1}}$

for $i = n$ downto 0 do $U_i=\frac{q_i-rigt_i\cdot U_{i+1}}{diag_i}$ There are $n + 2$ points; n = nb_points - 2.

Verifications!

It is unlikely to do all the calculations on the previous pages without making mistakes. It is therefore important to make several checks, which will allow errors to be detected.
Similarly, during the computer implementation, the following checks will make it possible to detect coding errors.

We want a translation of the control points to give the same translated curve.
Otherwise, the curve would be dependent on the choice of origin.
This implies the following 4 equalities:

1) $a_{i,1;1}+a_{i,1;2}+a_{i,1;3}+a_{i,1;4}=1$, verification that has already been done.
2) $a_{i,2;1}+a_{i,2;2}+a_{i,2;3}+a_{i,2;4}=0$
3) $a_{i,3;1}+a_{i,3;2}+a_{i,3;3}+a_{i,3;4}=0$
4) $a_{i,4;1}+a_{i,4;2}+a_{i,4;3}+a_{i,4;4}=0$

Useful rule for simplifications:
$h_{i+k;l}+h_{i+m;k-m}=t_{i+k+l}-t_{i+k}+t_{i+k}-t_{i+m}=h_{i+m;k+l-m2}$
$h_{i;2}=h_{i+1}+h_i;h_{i;3}=h_{i+2}+h_{i+1}+h_i$

2) gives

$$\frac{-3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i;1}-2h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}}{h_{i-1;3}\cdot h_{i;1}}$$
$$+\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}}{h_{i-1;3}\cdot h_{i;1}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_{i;1}\cdot h_{i-1;2}}$$
$$+\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_{i;1}\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}$$
$$+\frac{h_{i;2}\cdot h_{i;1}-h_{i;2}\cdot h_{i-1;1}-h_{i-1;1}\cdot h_{i;1}}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i;2}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_{i;1}\cdot h_{i-1;2}}+\frac{2h_{i-1;1}\cdot h_{i;1}-h_{i-1;1}^2}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{h_{i-1;1}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_{i;1}\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}$$
$$+\frac{h_{i;2}\cdot h_i-h_{i;2}\cdot h_{i-1}-h_{i-1}\cdot h_i+h_{i;2}\cdot h_{i-1;2}+2h_i\cdot h_{i-1}-h_{i-1}\cdot h_{i-1}+h_{i-1}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}$$
$$+\frac{h_{i;2}\cdot h_i-h_{i;2}\cdot h_{i-1}+h_{i;2}\cdot h_{i-1;2}+h_i\cdot h_{i-1}-h_{i-1}\cdot h_{i-1}+h_{i-1}\cdot h_{i-1;2}}{h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{(h_{i+1}+h_i)\cdot h_i-(h_{i+1}+h_i)\cdot h_{i-1}+(h_{i+1}+h_i)\cdot(h_i+h_{i-1})+h_i\cdot h_{i-1}-h_{i-1}\cdot h_{i-1}+h_{i-1}\cdot(h_i+h_{i-1})}{h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_{i;1}+h_{i-2;2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{h_{i+1}\cdot h_i+h_i\cdot h_i-h_{i+1}\cdot h_{i-1}-h_i\cdot h_{i-1}+h_{i+1}\cdot h_i+h_{i+1}\cdot h_{i-1}+h_i\cdot h_i+h_i\cdot h_{i-1}+h_i\cdot h_{i-1}-h_{i-1}\cdot h_{i-1}+h_i\cdot h_{i-1}+h_{i-1}\cdot h_{i-1}}{h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=-2\cdot\frac{h_i+h_{i-1}+h_{i-2}}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{2h_{i+1}\cdot h_i+2h_i\cdot h_i+2h_i\cdot h_{i-1}}{h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=\frac{-2\cdot(h_i+h_{i-1}+h_{i-2})\cdot h_{i-1;3}\cdot h_i+(2h_{i+1}\cdot h_i+2h_i\cdot h_i+2h_i\cdot h_{i-1})\cdot h_{i-2;3}}{h_{i-2;3}\cdot h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

$$=2\cdot\frac{-(h_i+h_{i-1}+h_{i-2})\cdot(h_{i+1}+h_i+h_{i-1})\cdot h_i+(h_{i+1}\cdot h_i+h_i\cdot h_i+h_i\cdot h_{i-1})\cdot(h_i+h_{i-1}+h_{i-2})}{h_{i-2;3}\cdot h_{i-1;3}\cdot h_i\cdot h_{i-1;2}}$$

= 0. It's long, but it's magic.
There is little chance of having this result with an error in the coefficients of the matrix!

3) $a_{i,3;1}+a_{i,3;2}+a_{i,3;3}+a_{i,3;4}=0$
“This verification is left as an exercise. We say that when we don't want to do it ourselves.

4) $a_{i,4;1}+a_{i,4;2}+a_{i,4;3}+a_{i,4;4}=0$, this verification is easy.

$$\frac{-1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}$$
$$+\frac{-1}{h_{i-1;3}\cdot h_{i-1;2}\cdot h_{i;1}}+\frac{-1}{h_{i-1;3}\cdot h_{i;2}\cdot h_{i;1}}+\frac{-1}{h_{i;3}\cdot h_{i;2}\cdot h_{i;1}}=0$$

Another approach is attempted in the following pages. It does not lead to a final result, but gives several possible verifications.

Let us look for a matrix *A* making the spline curve twice continuously differentiable, to find the matrix linked to a B-spline.

$$s_i(t)=\begin{pmatrix}1 & t-t_i & (t-t_i)^2 & (t-t_i)^3\end{pmatrix}\circ\begin{pmatrix}a_{1;1} & a_{1;2} & a_{1;3} & a_{1;4}\\ a_{2;1} & a_{2;2} & a_{2;3} & a_{2;4}\\ a_{3;1} & a_{3;2} & a_{3;3} & a_{3;4}\\ a_{4;1} & a_{4;2} & a_{4;3} & a_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$

$$s_i(t_{i+1})=\begin{pmatrix}1 & h_i & h_i^2 & h_i^3\end{pmatrix}\circ\begin{pmatrix}a_{1;1} & a_{1;2} & a_{1;3} & a_{1;4}\\ a_{2;1} & a_{2;2} & a_{2;3} & a_{2;4}\\ a_{3;1} & a_{3;2} & a_{3;3} & a_{3;4}\\ a_{4;1} & a_{4;2} & a_{4;3} & a_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}\mathrm{U}$$

$$s_{i+1}(t_{i+1})=\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\circ\begin{pmatrix}b_{1;1} & b_{1;2} & b_{1;3} & b_{1;4}\\ b_{2;1} & b_{2;2} & b_{2;3} & b_{2;4}\\ b_{3;1} & b_{3;2} & b_{3;3} & b_{3;4}\\ b_{4;1} & b_{4;2} & b_{4;3} & b_{4;4}\end{pmatrix}\circ\begin{pmatrix}U_i\\ U_{i+1}\\ U_{i+2}\\ U_{i+3}\end{pmatrix}$$

The $a_{j;k}$ will depend on the h_i.
The $bj_{;k}$ must depend on the h_i in the same way as the $a_{j;k}$ with the value of *i* shifted by +1.

For any U_i, the conditions to be met are:
$s_i(t_{i+1})=s_{i+1}(t_{i+1})$
$s_i(t_{i+1})=s_{i+1}(t_{i+1})$
$s_i''(t_{i+1})=s_{i+1}''(t_{i+1})$

$s_i(t_{i+1})=s_{i+1}(t_{i+1})$
The equality must be true all U_i, it imposes conditions, which breaks down into 5 equations:
$a_{1;1}+h_i\cdot a_{2;1}+h_i^2\cdot a_{3;1}+h_i^3\cdot a_{4;1}=0$
$a_{1;2}+h_i\cdot a_{2;2}+h_i^2\cdot a_{3;2}+h_i^3\cdot a_{4;2}=b_{1;1}$
$a_{1;3}+h_i\cdot a_{2;3}+h_i^2\cdot a_{3;3}+h_i^3\cdot a_{4;3}=b_{1;2}$
$a_{1;4}+h_i\cdot a_{2;4}+h_i^2\cdot a_{3;4}+h_i^3\cdot a_{4;4}=b_{1;3}$
$0=b_{1;4}$, so also : $a_{1;4}=0$ which has been seen previously.

The first condition is easily verified:

$$\frac{h_{i;1}\cdot h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}}+h_i\cdot\frac{-3h_{i;1}}{h_{i-2;3}\cdot h_{i-1;2}}+h_i^2\cdot\frac{3}{h_{i-2;3}\cdot h_{i-1;2}}+h_i^3\cdot\frac{-1}{h_{i-2;3}\cdot h_{i-1;2}\cdot h_{i;1}}$$

$$==\frac{h_i^2}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{-3h_i^2}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{3h_i^2}{h_{i-2;3}\cdot h_{i-1;2}}+\frac{-h_i^2}{h_{i-2;3}\cdot h_{i-1;2}}0$$

The other conditions are more complicated to verify.

$s'_i(t_{i+1}) = s'_{i+1}(t_{i+1})$
The equality must be true all U_i, it imposes conditions, which breaks down into 5 equations:
$a_{2;1} + 2h_i \cdot a_{3;1} + 3h_i^2 \cdot a_{4;1} = 0$
$a_{2;2} + 2h_i \cdot a_{3;2} + 3h_i^2 \cdot a_{4;2} = b_{2;1}$
$a_{2;3} + 2h_i \cdot a_{3;3} + 3h_i^2 \cdot a_{4;3} = b_{2;2}$
$a_{2;4} + 2h_i \cdot a_{3;4} + 3h_i^2 \cdot a_{4;4} = b_{2;3}$
$0 = b_{2;4}$, so also : $a_{2;4} = 0$which has been seen previously.

The first condition is easily verified:

$$\frac{-3h_{i;1}}{h_{i-2;3} \cdot h_{i-1;2}} + 2h_i \cdot \frac{3}{h_{i-2;3} \cdot h_{i-1;2}} + 3h_i^2 \cdot \frac{-1}{h_{i-2;3} \cdot h_{i-1;2} \cdot h_{i;1}}$$

$$== \frac{-3h_i}{h_{i-2;3} \cdot h_{i-1;2}} + \frac{6h_i}{h_{i-2;3} \cdot h_{i-1;2}} + \frac{-3h_i}{h_{i-2;3} \cdot h_{i-1;2}} 0$$

The other conditions are more complicated to verify.

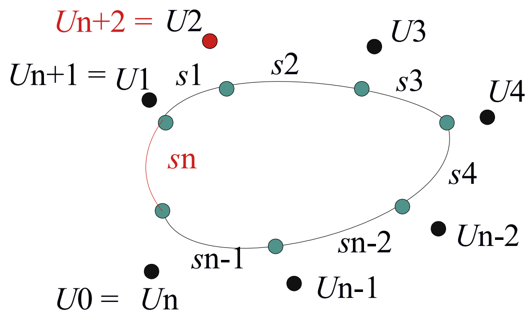
$s''_i(t_{i+1}) = s''_{i+1}(t_{i+1})$
The equality must be true all U_i, it imposes conditions, which breaks down into 5 equations:
$2a_{3;1} + 6h_i \cdot a_{4;1} = 0$
$2a_{3;2} + 6h_i \cdot a_{4;2} = 2b_{3;1}$
$2a_{3;3} + 6h_i \cdot a_{4;3} = 2b_{3;2}$
$2a_{3;4} + 6h_i \cdot a_{4;4} = 2b_{3;3}$
$0 = b_{3;4}$, so also : $a_{3;4} = 0$which has been seen previously.

The first condition is easily verified:

$$2\frac{3}{h_{i-2;3} \cdot h_{i-1;2}} + 6h_i \cdot \frac{-1}{h_{i-2;3} \cdot h_{i-1;2} \cdot h_{i;1}}$$

$$== \frac{2 \cdot 3}{h_{i-2;3} \cdot h_{i-1;2}} + \frac{-6}{h_{i-2;3} \cdot h_{i-1;2}} 0$$

The other conditions are more complicated to verify.
They can be verified numerically on examples, this can be useful for verifying the program.



Closed B-splines in the case of non-regular times t_i.
In the case of the drawing, $n = 7$

The goal is to add a segment s_n, so that the curve is closed.
To do this, we will add a point U_{n+2} and the segment s_n associated with the points U_{n+2}, U_{n+1}, U_n and U_{n-1}, and allow the position of the points U_{n+1} to be modified, and U_0.
We want this segment to end on U_0, ie C^1 and C^2 at U_0.

The conditions of continuity, of continuity of the derivative and of the second derivative are satisfied between s_{n-1} and s_n if we have as usual:
$s_n(t_n+\tau)=...$

On the other hand, three new conditions must be satisfied to have the continuity of the curve, the derivative and the second derivative.
It is therefore necessary to satisfy:
$s_n(t_{n+1}) = s_1(t_1)$, $(t_{n+1} = t_n + 1)$. So
$a_{n+1,1;1} \cdot U_n + a_{n+1,1;2} \cdot U_{n+1} + a_{n+1,1;3} \cdot U_{n+2} = a_{1,1;1} \cdot U_0 + a_{1,1;2} \cdot U_1 + a_{1,1;3} \cdot U_2$ and
$s'_n(t_{n+1}) = s'_1(t_1)$. So
$a_{n+1,2;1} \cdot U_n + a_{n+1,2;2} \cdot U_{n+1} + a_{n+1,2;3} \cdot U_{n+2} = a_{1,2;1} \cdot U_0 + a_{1,2;2} \cdot U_1 + a_{1,2;3} \cdot U_2$
$s''_n(t_{n+1}) = s''_1(t_1)$. So
$a_{n+1,3;1} \cdot U_n + a_{n+1,3;2} \cdot U_{n+1} + a_{n+1,3;3} \cdot U_{n+2} = a_{1,3;1} \cdot U_0 + a_{1,3;2} \cdot U_1 + a_{1,3;3} \cdot U_2$

In the case where one has chosen: $h_{n+i} = h_i$, with for example $h_0 = h_n$= distance between U_1 and U_n.

We check that they are satisfied if and only if:
$U_{n+2} = U_2$ and $U_{n+1} = U_1$ and $U_0 = U_n$

In conclusion, to close a B-spline, it is necessary to add a virtual point U_{n+2}, place it on point U_2 and place point U_0 on U_n and place point U_{n+1} on U_1.

The closed Math-spline linked to the points V_0 to V_{n+1} corresponding to the points U_0 to U_{n+1}, will give the same curve as the closed B-spline described above.
Note that the closed B-spline has one more point than the closed Math-spline!

The modification of the position of the points U_0 and U_{n+1} will modify the position of the points V_0, V_1, V_n and V_{n+1}, therefore the Math-spline curve. But it will not change the position of other points.
For a closed Math-spline, points V_0 and V_{n+1} are ignored.
These modifications of the points U_0 and U_{n+1} make it possible to close the curve while keeping it smooth, without disturbing the segments s_2 to s_{n-2}.

If a curve of a Math-spline is closed, then the corresponding B-spline will automatically have $U_{n+1} = U_1$ and $U_0 = U_n$. Closing the B-spline will give the same curve as the Math-spline.

Table of different cases of comparisons of B-splines and Math-splines

Let's compare various situations of links between B-splines and Math-splines

dT = 1

Math-spline \ B-spline	Ouvert	Fermé
Ouvert	B => M OK M => B OK	B => M OK M => B X
Fermé	B => M X M => B OK	B => M OK M => B OK

dT = distance entre points

Math-spline \ B-spline	Ouvert	Fermé
Ouvert	B => M OK M => B OK	B => M OK M => B X
Fermé	B => M X M => B OK	B => M OK M => B OK

"B => M" means: "move a point of the B-spline and update the points of the Math-spline"
"M => B" means: "move a point of the Math-spline and update the points of the B-spline"

OK means curves match after point move.
X means that the curves no longer necessarily match after the point move.

Case of open B-spline and closed Math-spline:

- it is normal that moving a point of the B-spline does not make it possible to have correspondence of the curves, because for a closed curve, points of the B- -spline must overlap, which is not imposed when the B-spline is open.
- moving a point of the Math-spline will force the overlapping points of the B-spline to give a curve superimposed on that of the Math-spline.

Case of closed B-spline and open Math-spline:

- it is normal that moving a point of the Math-spline does not make it possible to have correspondence of the curves, because for a closed curve, the first and the last point of the Math- -spline are imposed, which is not the case when the Math-spline is open.
- moving a point of the B-spline will force the positioning of the first and last point of the Math -spline so that it gives a curve superimposed on that of the B-spline.
  In fact, it will also force a positioning of the second and penultimate point.

In the case where both are open, there are no opposites which limit the adaptation of one curve to the other.

In the case where both are closed, both undergo constraints which allow the superposition of one curve on the other.

Appendix I, approximation of the second derivative of a function from 3 points.

To calculate a periodic spline curve, we approximated the second derivative at the start and end of the curve. The following justifies these approximations.

Let *f be* a C [2 function], hence twice continuously differentiable on the interval [t_m ; $_p$] _
$t_m < t_0 < t_p$ given.
With: $h_p = t_p - t_0$ and $h_m = t_0 - t_m$, both positive.
We have :

$$f(t_p) = f(t_0) + f'(t_0) \cdot h_p + f''(\tau_p) \cdot \frac{h_p}{2} \text{ where } \tau_p \in [t_0; t_p]$$

$$f(t_m) = f(t_0) - f'(t_0) \cdot h_m + f''(\tau_m) \cdot \frac{h_m}{2} \text{ where } \tau_m \in [t_m; t_0]$$

Let us show that the following expression is an approximation of the second derivative at t_0.

$$\frac{\frac{f(t_p) - f(t_0)}{h_p} - \frac{f(t_0) - f(t_m)}{h_m}}{h_p + h_m} \cdot 2 = ¿$$

$$\frac{2f'(t_0) + h_p \cdot f''(\tau_p) - 2f'(t_0) + h_m \cdot f''(\tau_m)}{h_p + h_m} = ¿$$

$$\frac{h_p \cdot f''(\tau_p) + h_m \cdot f''(\tau_m)}{h_p + h_m}$$ which is a weighted average of the second derivative before t_0 and after t_0.

This fraction is greater than $Min(f''(\tau_p); f''(\tau_m))$ and
less than $Max(f''(\tau_p); f''(\tau_m))$
Since the second derivative $f''(t)$ is continuous, the fraction is therefore equal to
$f''(\tau_O)$ for one $\tau_O \in [\tau_m; \tau_p]$.
It is therefore an approximation of the second derivative at t_0.

Annex II.
Checks that the curve obtained by B-spline is twice continuously differentiable.

Let's remember that :

$$s_i(t_i+\tau)=\frac{1}{6}\cdot\begin{pmatrix}1 & \tau & \tau^2 & \tau^3\end{pmatrix}\circ\begin{pmatrix}1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\circ\begin{pmatrix}U_{i-1}\\ U_i\\ U_{i+1}\\ U_{i+2}\end{pmatrix}$$ So

$$\begin{aligned}s(t_i+\tau)=\frac{1}{6}\cdot[&U_{i-1}+4U_i+U_{i+1}+\\ &\tau\cdot(-3U_{i-1}+3U_{i+1})+\\ &\tau^2\cdot(3U_{i-1}-6U_i+3U_{i+1})+\\ &\tau^3\cdot(-U_{i-1}+3U_i-3U_{i+1}+U_{i+2})]\end{aligned}$$

Let's calculate the derivative of s(t).

$$\begin{aligned}s'(t_i+\tau)=\frac{1}{6}\cdot[&-3U_{i-1}+3U_{i+1}+\\ &\tau\cdot(6U_{i-1}-12U_i+6U_{i+1})+\\ &\tau^2\cdot(-3U_{i-1}+9U_i-9U_{i+1}+3U_{i+2})]\end{aligned}$$

Let's calculate the second derivative of s(t).

$$\begin{aligned}s''(t_i+\tau)=&U_{i-1}-2U_i+U_{i+1}+\\ &\tau\cdot(-U_{i-1}+3U_i-3U_{i+1}+U_{i+2})]\end{aligned}$$

Verification of continuity in t_i.

$$s_i(t_{i+1})=\frac{1}{6}\cdot[U_i+4U_{i+1}+U_{i+2}]=s_{i+1}(t_{i+1})\text{okay}$$

Verification of the continuity of the derivative in t_i.

$$s'_i(t_{i+1})=\frac{1}{6}\cdot[-3U_i+3U_{i+2}]=s'_{i+1}(t_{i+1})\text{okay}$$

Checking the continuity of the second derivative at t_i.

$$s''_i(t_{i+1})=\frac{1}{6}\cdot[U_i-2U_{i+1}+U_{i+2}]=s''_{i+1}(t_{i+1})\text{okay}$$